Lesson 2-8 Fitting a Model to Data II

▶ **BIG IDEA** When a situation involves more than two variables that vary directly or inversely as each other, by pairing the dependent variable and each independent variable you can see how the variables fit into one *combined variation*.

Mental Math

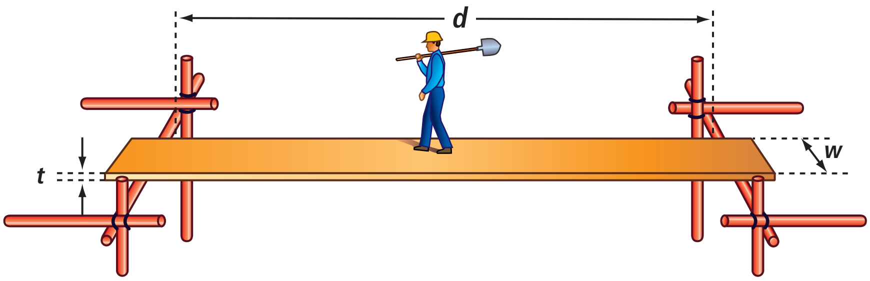
Find the point of intersection of the two lines.

a. $x = 2$ and $y = 7$

b. $x = 4$ and $y = 2x + 1.5$

c. $3x + y = 18$ and $y = 3$

So far in this chapter you have studied situations in which two quantities vary. In many real-world situations there are more than two variables. Consider the problem of determining the maximum weight that can be supported by a board.



Three quantities that influence the maximum weight M that a board can hold are the board's width w, thickness t, and the distance d between supports. The goal is to find an equation relating w, t, d, and the dependent variable M.

In Lesson 2-7, you used a graph in two dimensions to determine equations relating two variables. You cannot use a single graph to determine the equation representing this situation because there are four variables to be considered. However, by keeping *all but one* independent variable constant, you can investigate separately the relationship between the dependent variable M and that independent variable.

To obtain equations that model these relationships, you can use the converse of the Fundamental Theorem of Variation.

Converse of the Fundamental Theorem of Variation

a. If multiplying every x-value of a function by c results in multiplying the corresponding y-values by c^n, then y varies directly as the nth power of x, that is, $y = kx^n$.

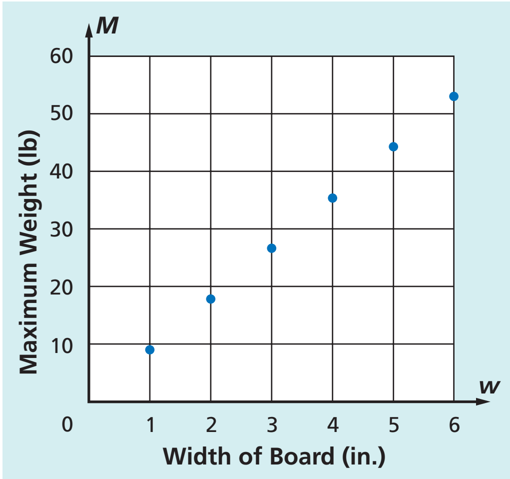
b. If multiplying every x-value of a function by c results in dividing the corresponding y-values by c^n, then y varies inversely as the nth power of x, that is, $y = \frac{k}{x^n}$.

Below is an explanation of how to find a model for the board problem. The data are made up, but the idea is not.

Maximum Weight as a Function of Width

Perry worked on this problem and found the model by working with the variables two at a time. First, Perry held the two independent variables d and t constant. He did this by placing supports at the ends of 12-foot long, 1.5-inch thick boards of different widths. He measured how much weight a board of a given width could support before it broke. Perry obtained the following data.

Width of Board w (in.)	1	2	3	4	5	6
Observed Maximum Weight M (lb)	9	18	26	36	44	53



The graph shows how the maximum weight M depends on the width w of the board. Because the points seem to lie on a line through the origin, Perry concluded that M varies directly as w.

GUIDED

Example 1

Verify that M varies directly as w.

Solution Assume $M = kw$. Select a data point (_?_, _?_) from the table and use it to find k.

_?_ = k _?_

_?_ = k

Use your k to write a variation formula for M as a function of w.

_?_ = _?_ • _?_

Use your formula to fill in values in the table predicted by this model.

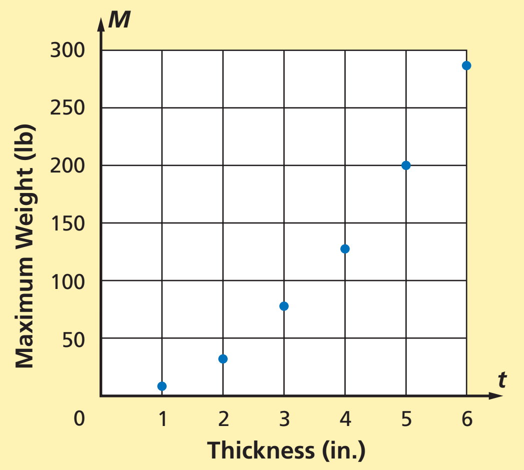
Width of Board w (in.)	1	2	3	4	5	6
Predicted Maximum Weight M (lb)	?	?	?	?	?	?

Your results should be almost identical to Perry's data, so this verifies that M varies ___?___ as ___?___.

Maximum Weight as a Function of Thickness

Perry then investigated the relationship between M and board thickness t. He held the distance d between supports constant at 12 feet, and the width w constant at 2 inches. He varied the thickness and measured the maximum weight that could be supported. The table below and graph at the right present his findings. On the graph, the points seem to lie on a parabola through the origin.

Thickness t (in.)	1	2	3	4	5	6
Predicted Maximum Weight M (lb)	8	32	72	128	200	288



Example 2

Use the Converse of the Fundamental Theorem of Variation to determine how M varies with t and write a variation equation for the situation.

Solution Select two data points (t_1, M_1) and (t_2, M_2) from the table such that t_2 is *double* the value of t_1.

$(t_1, M_1) = (2, 32)$ and $(t_2, M_2) = (4, 128)$.

$\frac{M_2}{M_1} = \frac{128}{32} = 4$, so, as t doubles, M is multiplied by $4 = 2^2$.

Select two other data points (t_3, M_3) and (t_4, M_4) from the table such that t_4 is *triple* the value of t_3.

$(t_3, M_3) = (1, 8)$ and $(t_4, M_4) = (3, 72)$

$\frac{M_4}{M_3} = \frac{72}{8} = 9$, so, as t triples, M is multiplied by $9 = 3^2$.

So, M varies directly as the square of t.

STOP QY1

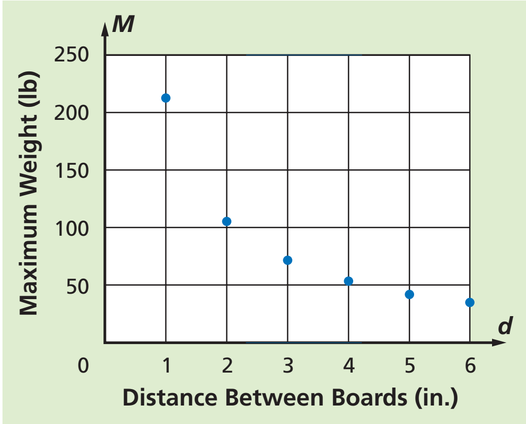
QY1

In Example 2, $M = kt^n$.
What is the value of n?
What is the value of k?

Maximum Weight as a Function of Distance between Supports

Next, Perry investigated the relationship between M and d by holding t and w constant. He chose boards 1.5 inches thick and 2 inches wide and measured the maximum weight that boards of different lengths would hold. Perry obtained the following data. The graph shows how M depends on d. It is not immediately clear whether M varies inversely as d or inversely as d^2.

Distance *d* (ft)	1	2	3	4	5	6
Observed Maximum Weight *M* (lb)	212	106	71	53	42	35



GUIDED

Example 3

Use the Converse of the Fundamental Theorem of Variation to determine how M varies with d. Then find the constant of variation k and use it to verify the relationship between M and d.

Solution Select three data points (d_1, M_1), (d_2, M_2), and (d_3, M_3) from the table such that d_2 is double the value of d_1 and d_3 is triple the value of d_1.

$(d_1, M_1) = (\underline{\ ?\ }, \underline{\ ?\ })$, $(d_2, M_2) = (\underline{\ ?\ }, \underline{\ ?\ })$,
and $(d_3, M_3) = (\underline{\ ?\ }, \underline{\ ?\ })$

$\frac{M_2}{M_1} = \underline{\ ?\ }$, so, as d doubles, M is divided by $\underline{\ ?\ }$.

$\frac{M_3}{M_1} = \underline{\ ?\ }$, so, as d triples, M is divided by $\underline{\ ?\ }$.

So, M varies $\underline{\ ?\ }$ as $\underline{\ ?\ }$, and $M = \frac{k}{?}$.

Now select a data point from the table and use it to find k.

$\underline{\ ?\ } = \frac{k}{?}$

$\underline{\ ?\ } = k$

Use k to write a variation formula for M as a function of d:

$\underline{\ ?\ }$.

Verify the relationship by using your formula to fill in the values of *M* in the table below.

Distance *d* (ft)	1	2	3	4	5	6
Predicted Maximum Weight *M* (lb)	?	?	?	?	?	?

Compare your results to Perry's data. Based on this comparison, you can confirm that M varies ___?___ as ___?___.

STOP QY2

> **QY2**
> Why are there different values for *k* in Examples 1, 2, and 3?

Summarizing the Results with a Single Model

Perry summarized his findings as follows:

M varies directly as *w* and the square of *t*, while *M* varies inversely as *d*.

These relationships can be expressed in the single formula

$$M = \frac{kwt^2}{d},$$

where *M* is in pounds, *w* and *t* are in inches, *d* is in feet, and *k* is the constant of variation. Notice that the independent variables that have direct-variation relationships with *M* are in the numerator, and the independent variable that has an inverse-variation relationship with *M* is in the denominator. The formula tells you that the greater the width and the thickness of the board, and the shorter the distance between supports, the stronger the board will be. This situation is an example of *combined variation*. You will learn more about combined variation in the next lesson.

Questions

COVERING THE IDEAS

1. Which variables in the Examples are the independent variables?
2. Why did it take three different tables to develop the single formula $M = \frac{kwt^2}{d}$?

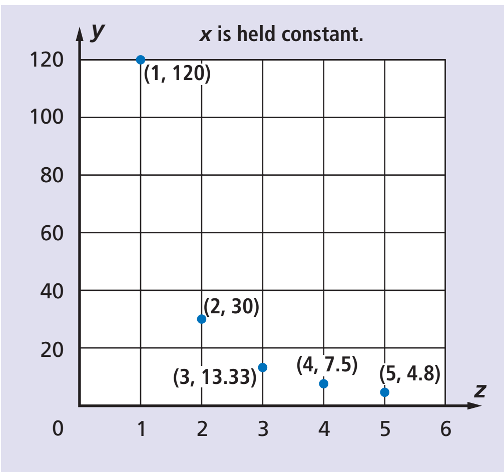
In 3–9, refer to the tables and graphs in this lesson.

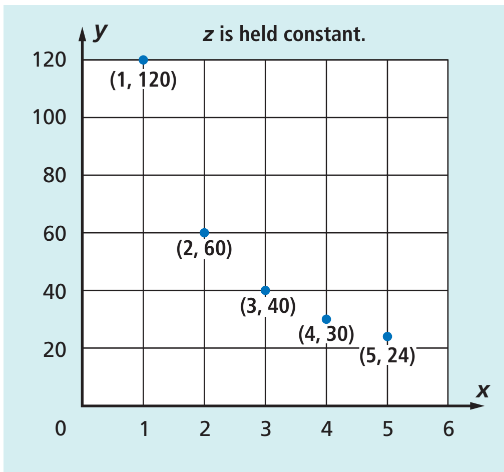
3. What is the maximum weight supported by a board 12 ft long, 2 in. wide, and 2 in. thick?
4. What is the maximum weight supported by a board 12 ft long, 6 in. wide, and 1.5 in. thick?
5. What is the shape of the graph relating M and w when t and d are held constant?
6. Refer to the data for M as a function of t. Find an equation to model the data when $d = 12$ ft and $w = 2$ in.
7. Use the Converse of the Fundamental Theorem of Variation to show that M does not vary inversely as d^2.
8. Suppose you run an experiment similar to the ones in the lesson and find that the maximum weight the board can hold is 50 pounds. Would you increase or decrease each of the following variables to increase the maximum weight the board could hold to 200 pounds?

   a. w b. t c. d
9. If M varies directly as a variable in the formula $M = \frac{kwt^2}{d}$, is that variable in the numerator or denominator of the expression on the right side of the formula?

APPLYING THE MATHEMATICS

10. **Multiple Choice** The two graphs below show the relationships between a dependent variable y and the independent variables x and z.





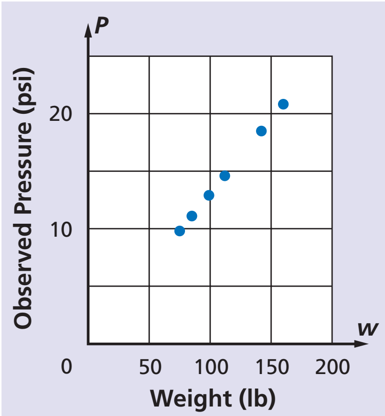
Which equation best models this situation?

A $y = \frac{kx^2}{z^2}$ B $y = \frac{kx}{z}$ C $y = \frac{k}{xz^2}$ D $y = \frac{k}{x^2z}$

11. Carrie was trying to determine how the pressure P that is exerted on the floor by the heel of a shoe depends on the heel width h and the weight w of the person wearing the shoe. She started by measuring the pressure (in psi) exerted by several people of different weights wearing a shoe with a heel 3 inches wide.

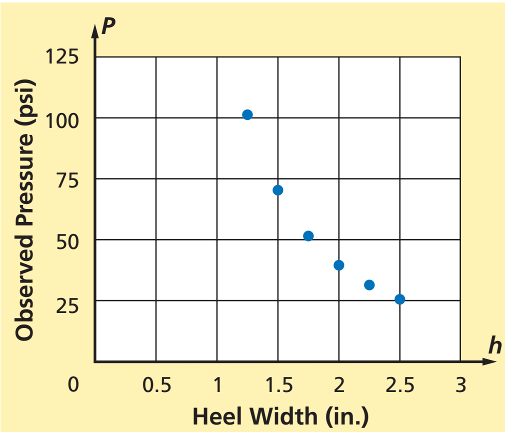
a. A table and graph of these data points are shown, using P as the dependent variable. How does the pressure exerted on the floor appear to relate to a person's weight? Use the Converse of the Fundamental Theorem of Variation to test your conclusion for two ordered pairs in the table.

Weight w (lb)	75	85	99	112	142	160
Observed Pressure P (psi)	9.8	11.1	12.9	14.6	18.5	20.8



b. Carrie then had her sister Candice, who weighs 135 pounds, try on shoes with different heel widths h, and she measured the pressure exerted. The data are summarized in the table and graph below, again using P as the dependent variable. Use the Converse of the Fundamental Theorem of Variation to determine if P varies inversely as h or as h^2.

Heel width h (in.)	1.25	1.5	1.75	2	2.25	2.5
Observed Pressure P (psi)	101.1	70.2	51.6	39.5	31.2	25.3

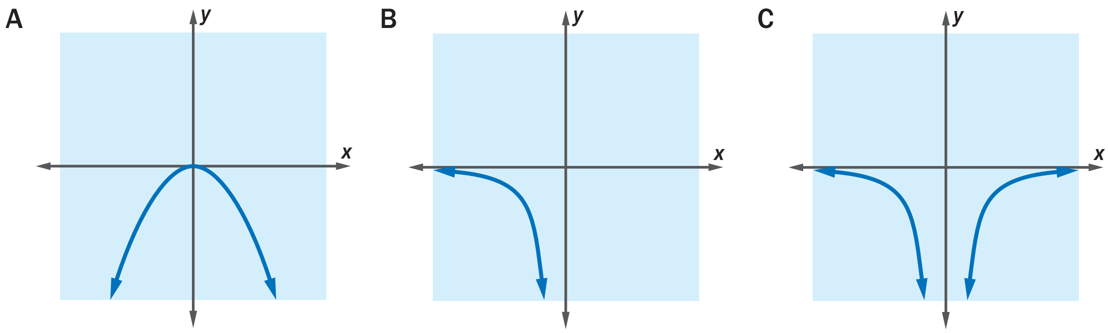


c. What amount of pressure would you expect to be exerted by Candice if she were wearing a 3-inch wide heel? Explain your reasoning.

d. Write an equation modeling the variation relationships between P, w, and h. Do not solve for k.

REVIEW

12. Consider $y = -\frac{7}{x^2}$ **(Lesson 2-6)**
   a. What real number is excluded from the domain of x?
   b. **Multiple Choice** Which could be the graph of the equation?



13. Identify all asymptotes of the graph of the equation. **(Lessons 2-4, 2-5, 2-6)**
   a. $y = \frac{7x}{4}$ b. $y = \pi x$ c. $y = -\frac{3}{x}$ d. $y = \frac{\frac{1}{2}}{x^2}$

14. **Fill in the Blanks** The graph of $y = \frac{5x}{3}$ is a __?__ with slope __?__. **(Lesson 2-4)**

15. Suppose r varies inversely as the cube of t. If r is 8 when t is 4, find r when $t = 9$. **(Lesson 2-2)**

16. If you buy x granola bars at y cents per bar, what is the total cost? **(Lesson 1-1)**

EXPLORATION

17. The ability of a board to support a weight also depends on the type of wood. That is, the constant of variation k in the formula $M = \frac{kwt^2}{d}$ depends on the type of wood.
   a. For a stronger kind of wood, is k larger or smaller?
   b. Do some research to find out which wood is strongest: oak, balsa, or pine.
   c. Suppose you have a 10-foot piece of pine that is 1 inch thick and 6 inches wide. If you wanted to cut a piece of oak that could hold the same amount of weight as the piece of pine, what do you know about the dimensions you should use for the piece of oak?

QY ANSWERS

1. $n = 2$; $k = 8$

2. Examples 1, 2, and 3 all describe different relationships, and as such they have different values of k.