Fitting a Model to Data I

▶ **BIG IDEA** If you determine from a particular set of data that *y* varies directly or inversely as *x*, you can graph the data to see what relationship is reasonable. Using that relationship, you can then substitute a known pair of values of *x* and *y* to obtain the constant *k* of variation, and compare what the variation equation predicts to the original data.

Mental Math

Emily is making a salad. The recipe calls for $5\frac{1}{2}$ cups of chopped lettuce. One head of lettuce yields about 3 cups of chopped lettuce.

a. How many heads of lettuce are needed for one recipe?

b. How many heads of lettuce does she need to double the recipe?

c. Heads of lettuce cost $2.53 each. How much will Emily spend on lettuce?

d. Emily can also buy 2-cup packages of chopped lettuce for $1.75 each. Would this be a better buy?
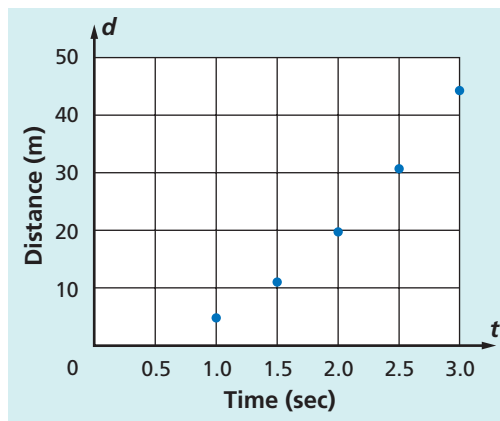
Recall from Chapter 1 that a mathematical model for a real situation is a description of that situation using the language and concepts of mathematics. Models can be created from collected data or from mathematical properties. Using data to create a mathematical model that describes those data is called *fitting a model to data*.

A Model of Direct Variation

In 1638, the Italian scientist Galileo Galilei (1564–1642) published *Dialogues Concerning Two New Sciences*, in which he first proposed the Law of Free-Falling Objects. This law states that near the surface of Earth, all heavier-than-air objects dropped from the same height fall to Earth in the same amount of time, assuming no resistance on the objects. (Aristotle, 1900 years earlier, wrote that heavier objects fall faster, and people believed Aristotle.) Equipment for timing free-falling bodies with sufficient precision did not exist, so Galileo tested his theory and developed a model by rolling objects down an inclined plane.

Today, scientists have more precise measuring techniques. For example, scientists can use slow-motion film to determine the distance an object in free fall travels over different periods of time. The table below gives the distance *d* in meters that a ball travels in *t* seconds after it is dropped from the top of a cliff. The ordered pairs in the table are graphed at the right.

Time in Air *t* (sec)	1	1.5	2	2.5	3
Distance Fallen *d* (m)	4.9	11.0	19.6	30.6	44.1

Because the distance the ball travels depends on the elapsed time, distance is the dependent variable and is placed on the vertical axis. How is the distance traveled related to time? Notice that distance increases as more time elapses; this implies direct variation. However, the points do not all lie on a straight line. The points suggest a direct-variation model of the form $y = kx^2$.

GUIDED

Example 1

Find a variation equation to describe the free-falling object data on the previous page.

Solution The shape of the graph suggests the formula $d = kt^2$. Substitute one of the ordered pairs into this formula to find k. The easiest pair to use is $(1, 4.9)$.

$$\underline{\quad?\quad} = k \ \underline{\quad?\quad}^2$$

$$k = \underline{\quad?\quad} \frac{m}{sec^2}$$

So, a variation equation describing the situation is $d = \underline{\quad?\quad} t^2$.

Check See how close the values predicted by the equation are to the values observed.

Time in Air t (sec)	1	1.5	2	2.5	3
Observed Distance Fallen d (m)	4.9	11.0	19.6	30.6	44.1
Predicted Distance Fallen d (m)	?	?	?	?	?

🛑 **QY1**

After fitting a model to a set of data, you can use the model to predict other data points that were not in the original set. The model $d = 4.9t^2$ predicts that the distance traveled in 6.5 seconds would be

$$d = (4.9)(6.5)^2 = 207.025.$$

That is, the distance would be about 207 meters.

🛑 **QY2**

A Model of Inverse Variation

Anna Lyzer and her sister Jenna went to a playground to investigate the validity of the Law of the Lever. Anna sat at a fixed point on one side of the seesaw while nine of her friends took turns sitting on the other side. When the seesaw was in balance, Jenna measured each person's distance d from the pivot and recorded their weight w. The results are shown on the next page.
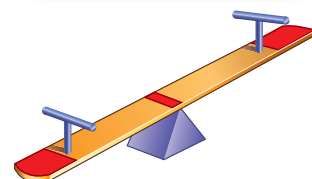
▶ **QY1**

If $k = 4.9$ m/sec^2 and $d = kt^2$, why is the unit for d meters?

▶ **QY2**

According to the model, approximately how far will an object fall in 8.25 seconds?

The shape of the graph shows that distance decreased as weight increased and suggests an inverse variation. You have seen graphs like this from two possible models: $d = \dfrac{k}{w^2}$ and $d = \dfrac{k}{w}$. The Law of the Lever says that $d = \dfrac{k}{w}$ is the more appropriate model, but how can the data tell us that? This Activity shows one way.

w (lb)	d (yd)
70	1.7
80	1.5
85	1.4
100	1.2
109	1.1
133	0.9
140	0.9
170	0.7
206	0.6
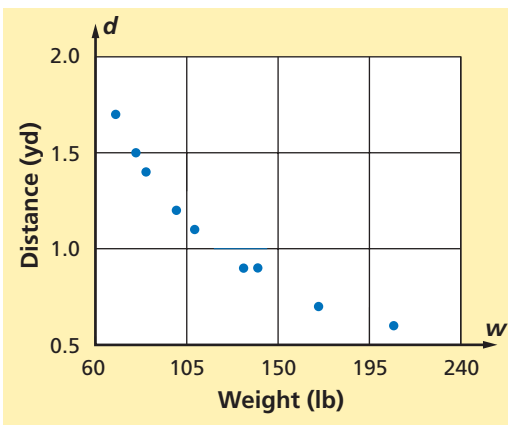
Activity

Step 1 First test $d = \dfrac{k}{w^2}$. To find k, select the point $(w, d) = (85, 1.4)$. Use the k-value you find to write an equation to model the situation.

Step 2 Use a spreadsheet. Enter the weights from the table above in the first column. Enter your variation equation from Step 1 at the top of the second column to generate a table of values. Compare the generated table to the values that Anna and Jenna observed. Do the values of d predicted by your model fit the observed data?

Step 3 Now test $d = \dfrac{k}{w}$. Use the same point $(w, d) = (85, 1.4)$ that you used in Step 1. Find k and write an equation to model this situation.

Step 4 Enter your variation equation from Step 3 at the top of the third column to generate a table of values. Compare the values predicted by this equation to the observed values that Anna and Jenna found.

Step 5 Do your findings confirm that $d = \dfrac{k}{w}$ is the better model? Why or why not?

Another way to approach this problem is to use the Fundamental Theorem of Variation.

Example 2

a. Use the Fundamental Theorem of Variation to determine an appropriate model that relates the weights and distances in Jenna's table of experimental values.

b. Predict the distance in yards that a person weighing 90 pounds must be from the pivot in order to balance Anna's weight.

Solution

a. When d varies inversely with the square of w, then if w is doubled, d is divided by 4. Find a pair of ordered pairs (w_1, d_1) and (w_2, d_2) where the ratio $\frac{w_2}{w_1}$ equals 2. One such pair of points is $(85, 1.4)$ and $(170, 0.7)$. Since $0.7 \neq \frac{1.4}{4}$, as the w-coordinate doubles, the d-coordinate is not divided by 4. Therefore, d does not vary inversely with the square of w. However, as the w-coordinate doubles, the d-coordinate is halved $(0.7 = \frac{1.4}{2})$. Therefore, the more appropriate model for these data is $d = \frac{k}{w}$. To find k, solve the formula for k and substitute an ordered pair into the equation.

$d = \frac{k}{w}$

$k = wd$ Multiply both sides by w.

$k = (85)(1.4) = 119$ Substitute the values of one ordered pair.

Using these two points, the data are modeled by

$d = \frac{119}{w}$.

b. If a person weighs 90 pounds, then $w = 90$. Using this model,

$d = \frac{119}{90} \approx 1.32$ yards.

So, sitting about 4 feet from the pivot will balance Anna.

STOP QY3

Questions

COVERING THE IDEAS

In 1–4, refer to the data about a free-falling object.

1. Describe in words the variation relationship between distance and time.

2. Use the model $d = 4.9t^2$ to predict the distance that a free-falling ball will fall in 4.5 seconds.

3. If a ball is dropped from a height of 500 meters, how many seconds will it take to reach the ground? (Ignore the effects of air resistance.)

Sky divers are not free-falling due to wind resistence.

4. Suppose a second ball is three times as heavy as the ball in Question 3. Compare the times it will take the balls to hit the ground.

In 5–7 refer to the Activity.

5. Use one of the data points in Jenna's table to show that $d = \frac{k}{w^2}$ is not a good model for the data.

6. Use the better model to predict the distance that a 180 lb person must be from the pivot in order to balance the seesaw with Anna.

7. How far from the pivot should a 20 lb baby sit to balance Anna? Does this seem possible in this situation?

APPLYING THE MATHEMATICS

8. Consider the equation $d = 4.9t^2$ where d is measured in meters and t is measured in seconds.

   a. Find the rate of change between the following pairs of points:
   (1, 4.9) and (1.5, 11.0)
   (1.5, 11.0) and (2, 19.6)
   (2, 19.6) and (2.5, 30.6)

   b. Is the rate of change a constant value?

   c. For this model, the rate of change is measured in what units?

9. Malcolm is blowing up a balloon. Refer to the table and graph, which give the number n of breaths he has blown into the balloon and the volume V of the balloon in cubic inches.

n	2	4	6	8	10	12
V	28.6	57.3	85.9	114.5	143.2	171.8
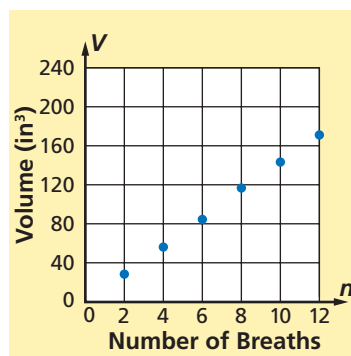
   a. **Multiple Choice** Which of the following equations is a good model for these data?

   i. $V = kn$ ii. $V = kn^2$

   iii. $V = \frac{k}{n}$ iv. $V = \frac{k}{n^2}$

   b. Find the constant k for your model.

   c. Use your model to predict the value of V when n is 14.

10. a. **Multiple Choice** Which formula best models the data graphed at the right?

   i. $L = ks$ ii. $L = ks^2$

   iii. $L = \frac{k}{s}$ iv. $L = \frac{k}{s^2}$

   b. Justify your answer to Part a.

   c. Predict the value of L when $s = 60$. Use the Fundamental Theorem of Variation to explain your prediction.
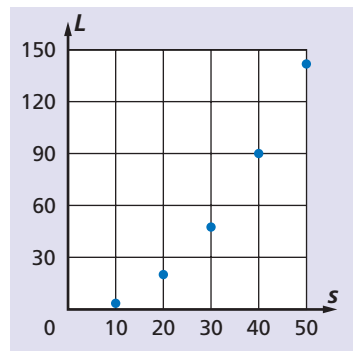
11. Refer to the table below and the graph at the right which show the intensity I of the sound, measured in decibels (dB), emitted from a 150-watt speaker at a distance d meters from the speaker.

d	1	2	3	4	5	6
I	11.9	3.0	1.3	0.75	0.48	0.33

a. **Multiple Choice** Which of the following equations is a good model for these data?

 i. $I = kd$ ii. $I = kd^2$

 iii. $I = \dfrac{k}{d}$ iv. $I = \dfrac{k}{d^2}$

b. Find the constant k for your model.

c. Alex looked at the graph and predicted that when d is 8, I is 0.62. Do you agree with him? Why or why not?

d. Use your model to predict the value of I when d is 10.

12. The deeper a diver goes below sea level, the greater the water pressure on the diver. To model this relationship, pressure data (in pounds per square inch, or psi) was recorded at various depths (in feet).

Depth d (ft)	0	15	20	25	30	35
Pressure p (psi)	0	6.5	8.6	10.8	12.9	15

a. Draw a graph of these data points. Let the depth d in feet be the independent variable and the pressure p in pounds per square inch be the dependent variable.

b. Write a variation function to model these data. Use one data point to calculate k and check the model with other data points.

c. Use your model to predict the value of p when $d = 40$.

d. Depth below sea level is often measured in *atmospheres*. Each atmosphere in fresh water equals 34 feet of vertical distance. For each additional atmosphere below sea level, there is an increase of 14.7 pounds per square inch (psi) of pressure on a diver. Explain whether or not this information is consistent with the model you have found in Parts b and c.
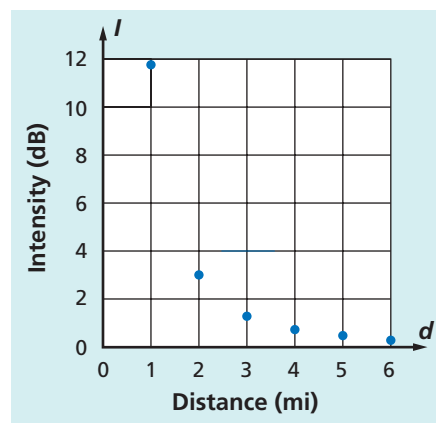
Deep sea divers must wear special suits to combat the enormous underwater pressure.

REVIEW

13. Pat weighs 2.5 times as much as her brother Matt. If they balance a seesaw, how do their distances from the pivot compare? Give sample weights and distances to support your answer. **(Lesson 2-6)**

14. Find the average rate of change between $x = 1$ and $x = 5$ for the function f with $f(x) = \frac{13}{x}$. **(Lesson 2-5)**

Multiple Choice In 15–18, match each graph to its equation. The scales on the axes are the same for all four graphs. **(Lessons 2-4, 2-5, 2-6)**

A $y = 2x$ **B** $y = \frac{2}{x}$ **C** $y = \frac{2}{x^2}$ **D** $y = -\frac{2}{x}$ **E** $y = -\frac{1}{2}x^2$

15. **16.** **17.** **18.**
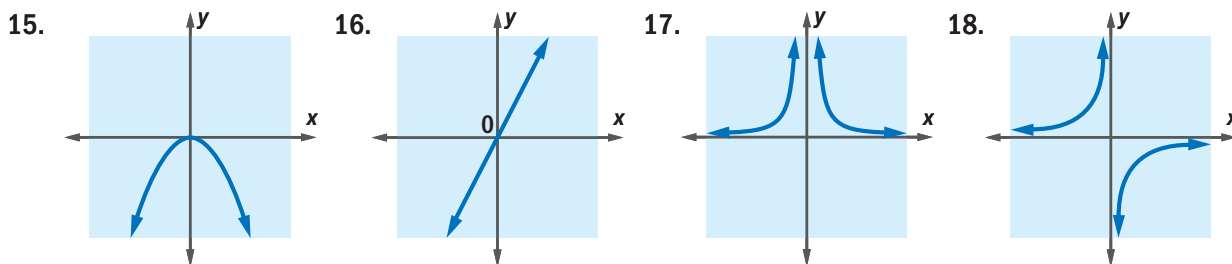
19. Suppose the value of x is tripled. How is the value of y changed if y is directly proportional to x^{23}? **(Lesson 2-3)**

20. Give the domain and range for each function graphed in Questions 15–18. **(Lesson 1-4, 2-1, 2-2)**

21. The table at the right provides data on the number of car sales in the U.S. from 1995 to 2004. **(Lessons 1-3, 1-5)**
  a. Let f be the function that maps the year y onto the number n of cars sold. Find $f(1999)$.
  b. For which year y is $f(y) = 8{,}104{,}000$?

22. Suppose V is the volume of a circular cylinder in cubic inches and r is the radius of the cylinder in inches. Let $h = \frac{V}{\pi r^2}$. What is h, and in what unit is h measured? **(Previous Course)**

EXPLORATION

23. Research Galileo's Law of Free-Falling Objects. Find out how he was able to use the results of rolling objects down an inclined plane as the basis for his conclusions on free-falling objects.

Year y	Car Sales $n = f(y)$ (thousands)
1995	8,635
1996	8,526
1997	8,272
1998	8,142
1999	8,698
2000	8,847
2001	8,423
2002	8,104
2003	7,610
2004	7,506

QY ANSWERS

1. The units multiply. $\frac{m}{sec^2} \cdot sec^2 = m$.

2. about 333.5 meters

3. about 0.74 yard; Yes, this distance is about half.