Lesson 2-6 The Graph of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$

BIG IDEA The graph of the set of points (x, y) satisfying $y = \frac{k}{x}$, with k constant, is a hyperbola with the x- and y-axes as asymptotes; the graph of the set of points (x, y) satisfying $y = \frac{k}{x^2}$ looks somewhat similiar but is closer to the axes.

Vocabulary

hyperbola
branches of a hyperbola
vertical asymptote
horizontal asymptote
discrete set
inverse-square curve

In previous lessons you explored the graphs of the equations of the direct-variation functions $y = kx$ and $y = kx^2$. Activity 1 explores the quite different graph of the equation of an inverse-variation function, $y = \frac{k}{x}$.

Activity 1

MATERIALS graphing utility

Step 1 Graph the function $y = \frac{10}{x}$ in a standard window. The resulting graph has points only in Quadrants I and III.

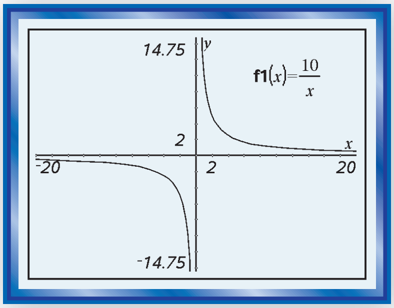
Step 2 Trace along the graph starting from a positive value of x and moving to the right. You can trace beyond the window edge if you want. It may also help if you zoom in on the graph for large values of x. Describe what happens to y as x increases in value.

Step 3 Now trace to the left starting from a negative value of x. Describe what happens to y as x decreases in value.

Step 4 Does the graph of $y = \frac{10}{x}$ ever intersect the x-axis? If it does, give the coordinates of any point(s) of intersection. If it does not, explain why not.

Step 5 Now trace closer and closer to $x = 0$ from both the positive and negative directions. Does the graph of $y = \frac{10}{x}$ ever intersect the y-axis? If it does, give the coordinates of any point(s) of intersection. If it does not, explain why not.

Step 6 Graph $y = \frac{k}{x}$ for a few nonzero values of k other than 10. Trace along the graphs past the edges of the standard window and near $x = 0$. Do all the graphs have the same behavior as x gets very far from 0 and very close to 0?



Mental Math

m∠1 = 72.
What is m∠2 if

a. ∠1 and ∠2 are supplementary?

b. ∠1 and ∠2 are the two acute angles in a right triangle?

c. ∠1 and ∠2 are vertical angles?

d. ∠1 and ∠2 form a linear pair?

The Graph of $y = \frac{k}{x}$

The graph of every function with an equation of the form $y = \frac{k}{x}$, where $k \neq 0$, is a **hyperbola**. In Activity 1 you graphed the hyperbola $y = \frac{10}{x}$. All hyperbolas with equation $y = \frac{k}{x}$ share some properties. For example, the functions they represent all have the same domain and range.

Example 1

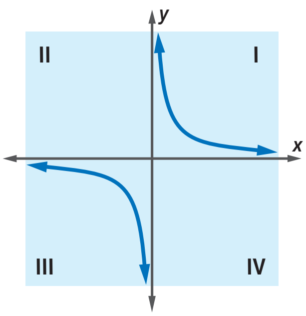
What are the domain and range of the function with equation $y = \frac{k}{x}$?

Solution To find the domain, think: What values can x have? You can substitute any number for x except 0 into the equation $y = \frac{k}{x}$.

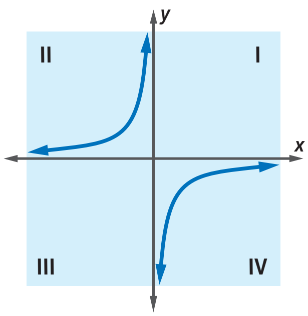
So, the domain of the function with equation $y = \frac{k}{x}$ is $\{x | x \neq 0\}$.

To find the range, think: What values can y have? Recall from the Activity that y can be positive or negative, large or small. But if $y = 0$, you would have $0 = \frac{k}{x}$, or $0 \cdot x = k$, which is impossible because, in the formula $y = \frac{k}{x}$ for a hyperbola, k cannot be zero. So, the range of the function with equation $y = \frac{k}{x}$ is $\{y | y \neq 0\}$.

The graphs in the $y = \frac{k}{x}$ family of curves share another property. Each hyperbola in this family consists of two separate parts, called **branches**. When $k > 0$, the branches of $y = \frac{k}{x}$ lie in the first and third quadrants; if $k < 0$, the branches lie in the second and fourth quadrants, as shown below.



$y = \frac{k}{x}$ when $k > 0$
Domain $\{x | x \neq 0\}$,
Range $\{y | y \neq 0\}$



$y = \frac{k}{x}$ when $k < 0$
Domain $\{x | x \neq 0\}$,
Range $\{y | y \neq 0\}$

Asymptotes

When $x = 0$, $y = \frac{k}{x}$ is undefined, so, the curve does not cross the y-axis. However, when x is *near* 0, the function is defined.

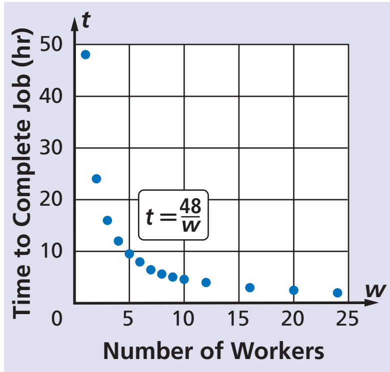
READING MATH

Hyperbola (a curve) and *hyperbole* (an exaggeration) come from the same Greek word *hyperbolē* meaning “excess.” Like *parabola*, *hyperbola* refers to a comparison of two distances; however, one of these distances is “in excess” of the other.

You found in Activity 1 that for $y = \frac{k}{x}$, $k > 0$, as x approaches zero from the positive direction, the y-value gets larger and larger. Similarly, as x approaches zero from the negative direction, the y-value gets smaller and smaller (more and more negative). As the values of x get closer and closer to a certain value a, if the values of the function get larger and larger or smaller and smaller, then the vertical line with equation $x = a$ is called a **vertical asymptote** of the graph of the function. The y-axis is a vertical asymptote to the graph of $y = \frac{k}{x}$, for $k \neq 0$.

Similarly, the x-axis is a **horizontal asymptote** of the curve $y = \frac{k}{x}$. As x gets very, very large (or very, very small), the value of y gets closer and closer to zero.

In some situations, only one branch of a hyperbola is relevant. For instance, recall the Condo Care Company example of Lesson 2-2. The time t it takes to complete the job and the number of workers w are related by the equation $t = \frac{48}{w}$. A table of values for this equation is shown below, along with a graph of the points in the table at the right. Because you cannot have a negative number of workers, there are no points in Quadrant III.



w	1	2	3	4	5	6	7	8	9	10	12	16	20	24
t	48	24	16	12	$9\frac{3}{5}$	8	$6\frac{6}{7}$	6	$5\frac{1}{3}$	$4\frac{4}{5}$	4	3	$2\frac{2}{5}$	2

In this example the function has a *discrete* domain, the set of positive integers. A **discrete set** is one in which there is a positive distance greater than some fixed amount between any two elements of the set. Because the number of workers is an integer, it does not make sense to connect the points of this graph. Thus, the graph consists of a set of discrete points on one branch of the hyperbola $y = \frac{48}{x}$.

STOP QY

▶ **QY**

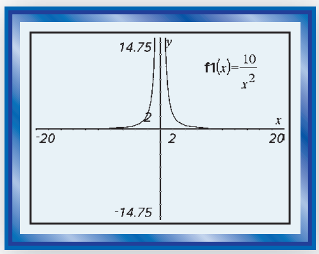
Identify the asymptotes of $y = \frac{48}{x}$.

Activity 2 explores the graphs of a different family of inverse-variation functions, those with equation $y = \frac{k}{x^2}$. We call the graph of such an inverse-variation function an **inverse-square curve**.

Activity 2

Step 1 Graph the function $y = \frac{10}{x^2}$ in a standard window. The resulting graph appears in Quadrants I and II. Why do you think this graph does not appear in Quadrants III or IV?

Step 2 Trace along the curve to a positive value of x. Continue tracing to the right. Describe what happens to y as x increases.



Step 3 Now trace to a negative value of x. Continue tracing to the left. Describe what happens to y as x decreases.

Step 4 Does the graph of $y = \frac{10}{x^2}$ ever intersect the x-axis? If it does, give the coordinates of the point(s) of intersection. If it does not, explain why not.

Step 5 Does the graph of $y = \frac{10}{x^2}$ ever intersect the y-axis? If it does, give the coordinates of the point(s) of intersection. If it does not, explain why not.

Step 6 Trace the graph close to $x = 0$ from both the positive direction and the negative direction. How does the graph behave near $x = 0$? Why does it behave this way?

The Graph of $y = \frac{k}{x^2}$

Example 2 examines the graph of $y = \frac{k}{x^2}$ for two values of k, one positive value and one negative value.

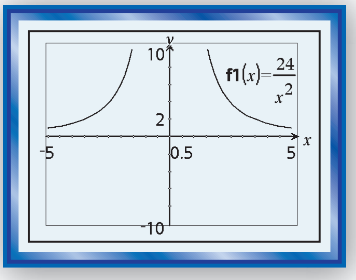
Example 2

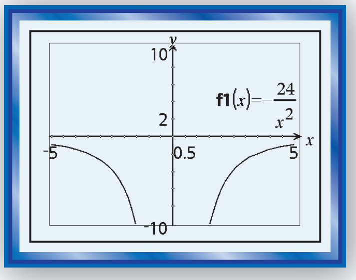
a. Graph $y = \frac{24}{x^2}$ in a window with $\{x| -5 \leq x \leq 5\}$ and $\{y| -10 \leq y \leq 10\}$. Sketch the graph on your paper. Then clear the screen and graph $y = -\frac{24}{x^2}$.

b. Describe the symmetry of each graph.

Solution

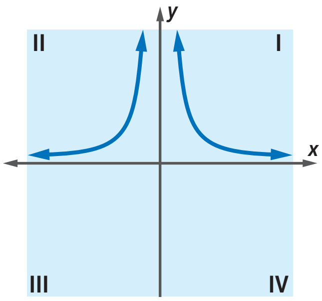
a. The graphs below were generated using a graphing utility.



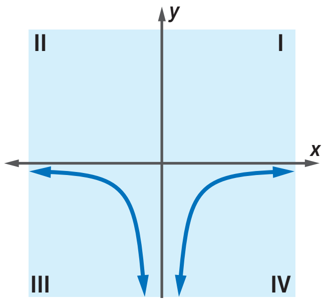


b. If either graph were reflected over the y-axis, the preimage and the image would coincide. So, each graph is symmetric about the y-axis.

Notice that the inverse-square curve, like a hyperbola, has two distinct branches. These two branches, however, do not form a hyperbola because the shape and relative position of the branches differ from a hyperbola. The domain of every inverse square function is $\{x| x \neq 0\}$. The range depends on the value of k. When k is positive, the range is $\{y| y > 0\}$. When k is negative, the range is $\{y| y < 0\}$. Graphs of the two types of inverse-square curves are shown on the next page.



$y = \frac{k}{x^2}$ for $k > 0$
Domain $\{x | x \neq 0\}$
Range $\{y | y \neq 0\}$



$y = \frac{k}{x^2}$ for $k < 0$
Domain $\{x | x \neq 0\}$
Range $\{y | y < 0\}$

Notice that the graph of $y = \frac{k}{x^2}$ has the same vertical and horizontal asymptotes as the graph of $y = \frac{k}{x}$.

In general, asymptotes may be vertical, horizontal, or oblique, but not all graphs have asymptotes. For example, the parabola $y = kx^2$ does not have any asymptotes. When a graph has an asymptote, that asymptote is *not* part of the graph.

Questions

COVERING THE IDEAS

1. Why is 0 not in the domain of the functions with equations $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$?

2. For what values of k is the range of the function $y = \frac{k}{x^2}$ the set $\{y | y < 0\}$?

3. Suppose $k \neq 0$. What shape is the graph of $y = \frac{k}{x}$?

In 4 and 5, consider the function f with equation $f(x) = \frac{15}{x}$.

4. State the domain and range of f.

5. **a.** Evaluate $f(-10)$ and $f(15)$.
   b. Graph f in the window $-12 \leq x \leq 12$ and $-12 \leq y \leq 12$ and sketch the graph.

6. Refer to the Condo Care Company situation and the graph of $t = \frac{48}{w}$ in this lesson. Why is the domain discrete?

7. Suppose $k \neq 0$. Is the graph of the equation symmetric to the y-axis?
   a. $y = \frac{k}{x}$
   b. $y = \frac{k}{x^2}$

8. In which quadrants are the branches of $y = \frac{k}{x^2}$
   a. if k is positive?
   b. if k is negative?

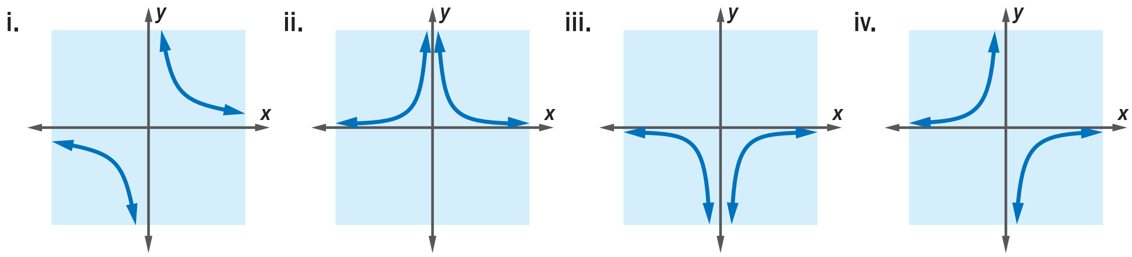
9. What happens to the y-coordinates of the graph of $y = \frac{21}{x}$ when x is positive and is getting closer and closer to 0?

10. Identify the asymptotes of the graph of $y = \frac{4}{x}$.

APPLYING THE MATHEMATICS

11. Match each equation with the proper graph.

    a. $y = \frac{5}{x^2}$ b. $y = -\frac{10}{x^2}$ c. $y = \frac{20}{x}$ d. $y = -\frac{5}{x}$



12. Compare and contrast the graphs of $y = -\frac{k}{x^2}$ and $y = \frac{k}{x^2}$ when $k = \frac{1}{10}$.

13. *The Doorbell Rang*, by Pat Hutchins (1986), tells the story of 12 cookies to be divided among an increasing number of people. The story begins with 2 kids who want to share the cookies. Then another kid comes to the door and they must divide the cookies 3 ways, then 4, and so on.

    a. The table at the right shows some values of the function described in the story. Fill in the missing values.

    b. Write an equation for a function that contains the coordinate pairs in the table. Explain the meaning of your variables.

    c. Explain why this function has a discrete domain.

Number of Kids	Number of Cookies Each Kid Gets
1	12
2	6
3	?
4	3
5	$\frac{12}{5} = 2.4$
6	?
7	$\frac{12}{7} \approx 1.7$
8	?
9	$\frac{12}{9} \approx 1.3$
10	?

14. Quentin is on a seesaw. He weighs 120 pounds and is sitting 5 feet from the pivot. Recall that the Law of the Lever is $d = \frac{k}{w}$, where d is distance from the pivot, w is weight, and k is a constant of variation.

    a. Find k and write a variation equation for this situation.

    b. Make a table of 10 different weights and their distances from the pivot that would balance Quentin.

    c. Plot your values from Part b.

    d. Is the domain for the context discrete? Explain why or why not.

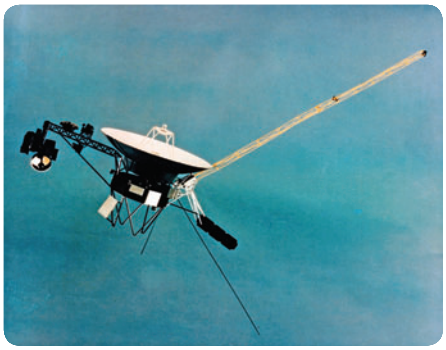
15. Consider the functions f and g described by the equations $f(x) = \frac{36}{x}$ and $g(x) = \frac{36}{x^2}$.

a. Use a graphing utility to graph both functions on the same axes, with the window $-12 \le x \le 12$ and $-60 \le y \le 60$. Describe some similarities and differences between the graphs.

b. Find and compare the average rates of change of each function between $x = 1$ and $x = 6$.

c. Find and compare the average rates of change of each function between $x = 6$ and $x = 12$.

16. According to Newton's Law of Gravitation, the force of gravity between two objects obeys an inverse-square law with respect to the distance between the two objects. For example, the force of gravity between Earth and the Voyager 1 space probe launched in 1977 is approximately $F = \frac{1.56 \cdot 10^{12}}{d^2}$, where d is the distance between Earth and Voyager 1 in kilometers, and F is the force in newtons.



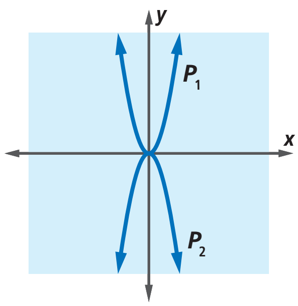
The space probe Voyager 1

a. In the summer of 2002, Voyager 1 was approximately $1.3 \cdot 10^{10}$ km from Earth. What was the force of gravity between Voyager 1 and Earth at that time?

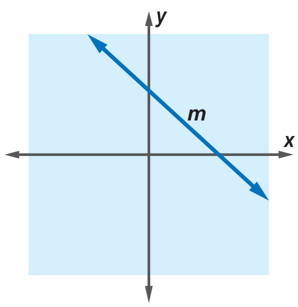
b. Will Voyager 1 ever completely escape Earth's gravity? Explain your answer.

REVIEW

17. In the graph at the right, parabolas P_1 and P_2 are reflection images of each other over the x-axis. If parabola P_1 has equation $y = 3x^2$, what is an equation for parabola P_2? **(Lesson 2-5)**



18. Does the line m graphed below at the right illustrate an example of direct variation, inverse variation, or neither? Justify your answer. **(Lesson 2-4)**



19. Dominique, a scuba diver 99 feet below the water's surface, inflates a balloon until it has a diameter of 6 inches. The air in the balloon expands as Dominique ascends. When Dominique reaches the surface, the balloon has a diameter of 9.5 inches. **(Lesson 2-3)**

a. Write the ratio of the radius of the balloon at the surface to the radius 99 feet below the surface.

b. The volume of the balloon at the surface is how many times the volume of the balloon at 99 feet?

c. Explain how your answer to Part a can be used to answer Part b.

Multiple Choice In 20 and 21, which could be a formula for the nth term of the sequence? **(Lesson 1-8)**

20. 2, 4, 8, 16, 32, …

A $t_n = 2n$ B $t_n = n^2$ C $t_n = 2^n$

21. 2, 9, 28, 65, 126, …

A $t_n = 7n - 5$ B $t_n = 7n^2 - 2$ C $t_n = n^3 + 1$

22. Solve for x: $y = -\frac{1}{\pi}x$. **(Lesson 1-7)**

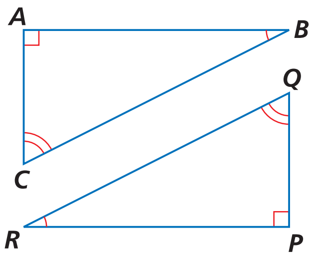
23. Bob decided to bake a frozen pizza for dinner. It took 10 minutes to preheat the oven to 425° from room temperature of 75°. The pizza baked for 17 minutes, and then it took another 30 minutes for the oven to return to room temperature after Bob turned it off. **(Lessons 1-4, 1-2)**

a. Identify the independent and dependent variables in this situation.

b. Sketch a graph of the situation. Label the axes and explain the meaning of your variables.

c. Identify the domain and range of the function mapping time onto the temperature of the oven.

24. Multiple Choice The relationship between $\triangle ABC$ and $\triangle PRQ$, as shown below, is best described by which of the following? **(Previous Course)**



A congruent

B similar

C both congruent and similar

D neither congruent nor similar

EXPLORATION

25. Use a graphing utility to graph $y = \frac{10}{x^3}$, $y = \frac{10}{x^4}$, and $y = \frac{10}{x^5}$. What pattern do you notice? What generalization can you make about the value of n and the graph of $y = \frac{10}{x^n}$? Is the same generalization true for every function with equation $y = \frac{k}{x^n}$, $k \neq 0$?

QY ANSWERS

x-axis and y-axis