Lesson

2-5 The Graph of $y = kx^2$

Vocabulary

parabola

vertex of a parabola

reflection-symmetric

line of symmetry

BIG IDEA The graph of the set of points (x, y) satisfying $y = kx^2$, with k constant, is a parabola with vertex at the origin and containing the point $(1, k)$.

Mental Math

Suppose that packs of gum cost 39 cents each.

a. What will be your change if you pay for two packs with a $1 bill?

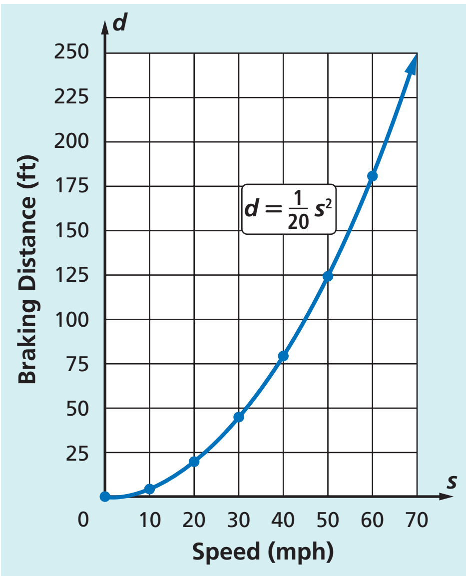
b. What will be your change if you pay for eight packs with a $5 bill?

c. What will be your change if you pay for twenty packs with a $10 bill?

d. Now suppose you want 20 packs and the store is having a buy-three-packs, get-one-free sale. What will be your change if you pay with a $10 bill?

In Lesson 1-4, you studied functions whose values gave the stopping distances of a vehicle at various speeds. Stopping distance includes reaction time, braking distance, and other factors. Braking distance is simply the distance needed to stop a vehicle after applying the brake. Using mathematics you will study in calculus, it can be proved that braking distance varies directly with the square of a vehicle's speed. In Question 16 of Lesson 2-1, you found that the formula $d = \frac{1}{20}s^2$ describes this relation for a typical car. The table and graph below represent this relation.

Speed s (mph)	Braking Distance d (ft)
0	0
10	5
20	20
30	45
40	80
50	125
60	180
70	245



Rates of Change

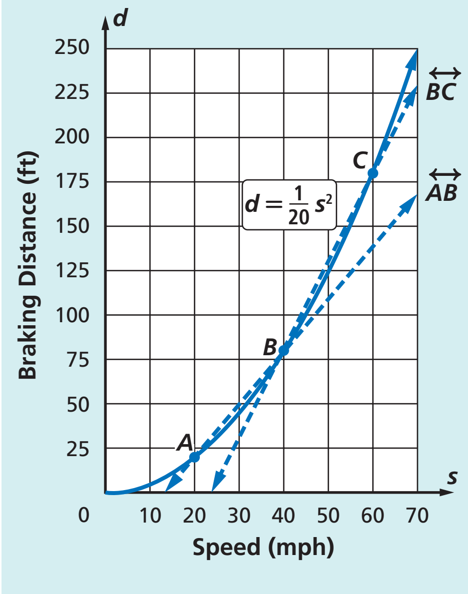
The points on the graph above do not lie on a straight line. You can verify this by calculating the slopes of the lines through different pairs of points on the graph and seeing that they are not equal.

Example 1

Find the following rates of change and explain what each means in terms of braking distance.

a. r_1, **the rate of change from (20, 20) to (40, 80)**

b. r_2, **the rate of change from (40, 80) to (60, 180)**



Solution

a. Use the definition of slope.

$$r_1 = \frac{80 \text{ ft} - 20 \text{ ft}}{40 \text{ mph} - 20 \text{ mph}} = \frac{60 \text{ ft}}{20 \text{ mph}} = \frac{3 \text{ ft}}{\text{mph}}$$

This means that, on average, when driving between 20 mph and 40 mph, for every increase of 1 mph in speed, you need 3 more feet to stop your car.

b. Similarly, $r_2 = \frac{180 \text{ ft} - 80 \text{ ft}}{60 \text{ mph} - 40 \text{ mph}} = \frac{100 \text{ ft}}{20 \text{ mph}} = 5\,\frac{\text{ft}}{\text{mph}}$.
So on average, between $s = 40$ and $s = 60$, for every change of 1 mph (the horizontal unit), there is a change of 5 feet of braking distance (the vertical unit).

Check Look at the points on the graph. Let $A = (20, 40)$, $B = (40, 80)$, and $C = (60, 180)$. Is $\overleftrightarrow{BC}$ steeper than $\overleftrightarrow{AB}$? Yes, it is.

STOP **QY1**

> **QY1**
>
> Calculate the average rate of change between $A = (20, 40)$ and $C = (60, 180)$ in Example 1 and explain what it means.

The rate of change between different pairs of points on the graph of $d = \frac{1}{20}s^2$ is not constant. Two conclusions can be drawn:

1. The graph of $d = \frac{1}{20}s^2$ is not a line.
2. A single number does not describe the slope of the whole graph.

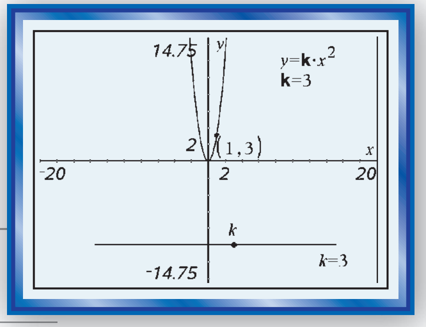
Notice that the slope is larger where the graph is steeper, meaning the braking distance increases more and more rapidly as the speed increases.

The equation $d = \frac{1}{20}s^2$ represents a direct-variation function of the form $y = kx^2$. All graphs of equations of this form share some properties. You should be able to sketch graphs of any equation of this form.

Activity

MATERIALS CAS, graphing calculator, or variation graph application

In this Activity, you will explore graphs of the family of curves $y = kx^2$.



Step 1 Consider the family of curves $y = kx^2$. The graph has point $A = (1, k)$ on it. Verify that the coordinates for point A are correct by substituting $x = 1$ into $y = kx^2$.

Step 2 Grab and drag point A to vary the shape of the graph. Since point A has y-coordinate k, the value of k *varies* as you drag A.

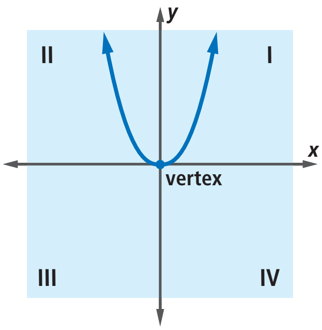
Step 3 When $k > 0$, and k increases in value, how does the graph of $y = kx^2$ change?

Step 4 When $k = 0$, describe the appearance of the graph $y = kx^2$. What are the coordinates of A?

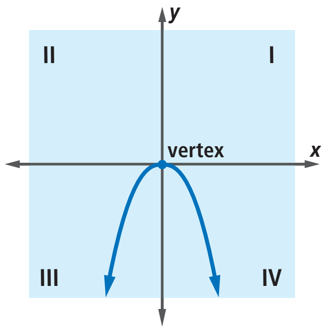
Step 5 When $k < 0$,and k decreases in value, how does the graph of $y = kx^2$ change? How is the graph of $y = kx^2$ different when k is negative from when k is positive?

The Graph of $y = kx^2$

All of the graphs in the family of curves with the equation $y = kx^2$ with $k \neq 0$ are **parabolas**. No matter what value is chosen for k, the parabola passes through the point $(0, 0)$. This point is the parabola's **vertex**. Further, each parabola in this family coincides with its reflection image over the y-axis. Thus, each parabola is **reflection-symmetric**, and the y-axis is called the **line of symmetry**.



$y = kx^2$, $k > 0$
parabola opens up
Domain: the set of all reals
Range: the set of nonnegative reals



$y = kx^2$, $k < 0$
parabola opens down
Domain: the set of all reals
Range: the set of nonpositive reals

In general, the domain of the function with equation $y = kx^2$ is the set of all real numbers. As you saw in the Activity, when $k > 0$, the range is the set of nonnegative real numbers, and the graph of the function is a parabola that *opens up*. That is, the vertex of the parabola is its *minimum* point. When $k < 0$, the range is the set of nonpositive real numbers and the corresponding parabola *opens down*. That is, the vertex of the parabola is its *maximum* point.

READING MATH

Parabola (a curve) and *parable* (a story) come from the same Greek word *parabolē* meaning "comparison." The *parabola* is named for a comparison of two distances that are equal.

STOP QY2

The equation $y = kx^2$ is not the only equation whose graph is a parabola. You will study other equations leading to parabolas in Chapter 6.

QY2

Without using a graphing utility, sketch rough graphs of $y = -2x^2$ and $y = 2x^2$ on the same set of axes.

Questions

COVERING THE IDEAS

In 1 and 2, refer to the formula $d = \frac{1}{20}s^2$ relating speed and braking distance.

1. Find the average rate of change between each pair of points and explain what it means.
   a. (10, 5) and (20, 20)
   b. (50, 125) and (60, 180)
   c. (a, b) and (c, d)
2. Your answers to Questions 1a and 1b should be different numbers. What does that tell you about the graph of $d = \frac{1}{20}s^2$?
3. Name the type of curve that results from graphing $y = kx^2$ $(k \neq 0)$.
4. Explain what it means to say that the graph of $y = kx^2$ is symmetric to the y-axis.
5. Suppose $k < 0$. State the domain and range of the function $f: x \rightarrow kx^2$.
6. Refer to the Activity. Describe how the graph of $y = kx^2$ changes in each situation.
   a. k increases from 1 to 10
   b. k decreases from 1 to 0
7. For what values of k does the graph of $y = kx^2$
   a. open up?
   b. open down?

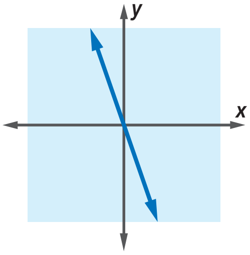
In 8 and 9, one variable varies directly as another.
   a. Name the variables.
   b. Name the constant of variation.

8. $S.A. = 4\pi r^2$ (surface area of a sphere)
9. $A = \frac{s^2}{4}\sqrt{3}$ (area of an equilateral triangle)

APPLYING THE MATHEMATICS

10. **Matching** Match each equation with the proper graph. Assume each graph has the same scale.

a. $y = \frac{1}{4}x^2$ **b.** $y = -3x$ **c.** $y = -2x^2$ **d.** $y = 2x^2$

i.


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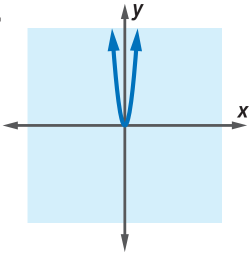

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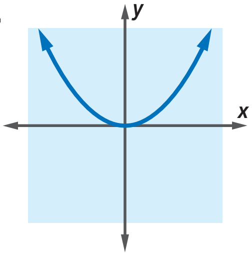

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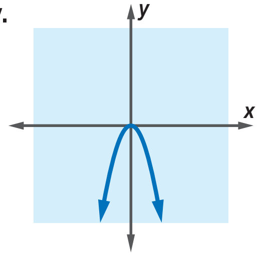


11. **a.** Explain why the point (0, 0) is on the graph of $y = kx^2$ for all values of k.

b. Explain how to use the point $(1, f(1))$ to determine the value of k if $f(x) = kx^2$.

12. On some cameras, you can control the diameter of the aperture, or the opening through which light passes to the film, by setting an f-stop. You also control the *exposure*, or the length of time the aperture is open, to ensure that the proper amount of light reaches the film. Let E be the exposure (in seconds) and f be the f-stop of a camera. Suppose you know that E varies directly as either f or f^2, and the constant of variation is $\frac{1}{6000}$.

a. Assume that E varies directly as f, so $E = \frac{1}{6000}f$. Make a table of ordered pairs for this function. For values of f use 1, 1.4, 2, 2.8, 4, and 5.6.

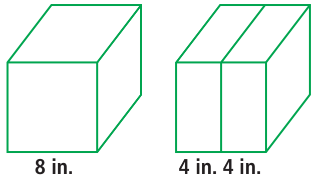
b. Using a special camera, you set an f-stop of 4.8. Use your table to find two values close to the required exposure. Average the two values to make an estimate of E for $f = 4.8$.

c. Graph the ordered pairs from the table on a coordinate grid and connect the points with a smooth curve. Use the graph to estimate the required exposure.

d. Using the variation equation, determine the actual exposure required for an f-stop of 4.8. How do the three values you found for the exposure compare?

e. Repeat Parts a–d, but this time assume that E varies directly as f^2, so $E = \frac{1}{6000}f^2$. How do the three values you found for the exposure compare? How does the shape of the graph affect the accuracy of these methods for estimating E?

13. The maximum load a beam can safely support varies directly as its width. A contractor suggests replacing an 8''-wide beam that can hold 6000 pounds with two 4''-wide beams made of the same material as the 8'' beam, one next to the other. Can the two 4'' beams replace a single 8'' beam safely? Justify your answer.



14. Suppose (3, 5) is on the graph of $y = kx^2$. Is this sufficient information to determine k? If so, find k. If not, why not?

REVIEW

15. Consider the following four equations. **(Lesson 2-4, Previous Course)**

$y = 3x$ $\qquad$ $y = \frac{1}{3}x$ $\qquad$ $y = -\frac{1}{3}x$ $\qquad$ $y = -3x$

   a. Graph the equations on one set of axes.

   b. Find the slope of each line.

   c. Give the equations of two lines that appear to be perpendicular.

16. Find the slope of a submarine dive if the submarine drops 1,200 feet while moving forward 4,000 feet. **(Lesson 2-4)**

17. Consider the sequence t defined by $t_n = 3n^2 - 2n + 2$. **(Lessons 1-2, 1-5, 1-8)**

   a. What is the domain of the sequence?

   b. Use your CAS to generate a table containing the first forty terms of this sequence. Write down the last five.

18. Onida's annual salary is $42,000. She gets an increase of 6% at the end of each year. The formula $a_n = 42{,}000(1.06)^n$ gives Onida's salary a_n at the end of the nth year. **(Lesson 1-8)**

   a. Find Onida's salary at the end of each year for the first five years.

   b. What would Onida's salary be after 20 years with this company?

19. a. Write a letter of the alphabet that has exactly one line of symmetry.

   b. Write a letter of the alphabet that has exactly two lines of symmetry. **(Previous Course)**

EXPLORATION

20. The f-stop settings on a camera are related to the diameter of the aperture of the camera. Increasing the f-stop decreases the diameter of the aperture. Research to find out the relationship between these two variables.

QY ANSWERS

1. $3.5 \frac{\text{ft}}{\text{mph}}$; when you are driving between 20 mph and 60 mph, on average, for every increase of 1 mph, there is an increase of 3.5 ft in braking distance.

2.