Lesson 2-4 The Graph of $y = kx$

Vocabulary

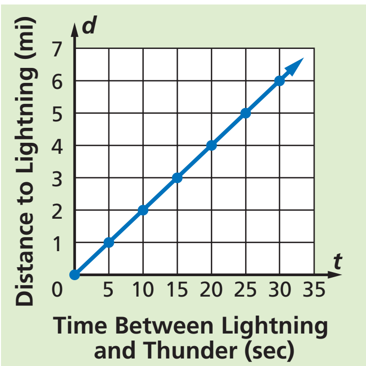
rate of change

slope

BIG IDEA The graph of the set of points (x, y) satisfying $y = kx$, with k constant, is the line containing the origin and having slope k.

Recall from the Questions in Lesson 2-1 that the distance d you are from a lightning strike varies directly with the time t elapsed between seeing the lightning and hearing the thunder. The formula $d = \frac{1}{5}t$ describes this situation for the values given in that lesson. This direct-variation function can also be represented graphically. A table and a graph for the equation $d = \frac{1}{5}t$ are shown here.

Time t (sec)	0	5	10	15	20	25	30
Distance d (mi)	0	1	2	3	4	5	6



Mental Math

Consider this set of numbers: −17, 512, $-\sqrt{2}$, $\frac{7}{11}$, 0.0145.

a. Find the median of this set.

b. Which numbers in the set are irrational?

c. Which numbers are rational but not integers?

d. Which numbers are not in the domain of f if $f(x) = \sqrt{4 - x^2}$?

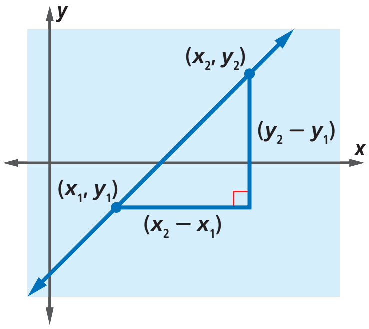
Note that the domain of this function is the set of nonnegative real numbers, and the range is also the set of nonnegative real numbers. When all real-world solutions to the equation $d = \frac{1}{5}t$ are plotted in the coordinate plane, the graph is a ray starting at the origin and passing through the first quadrant. There are no points on the graph of $d = \frac{1}{5}t$ in any other quadrants.

STOP **QY1**

> **QY1**
>
> Explain why there could be no real-world solutions in Quadrants II, III, or IV.

The Slope of a Line

Recall that the steepness of a line is measured by a number called the *slope*. The slope of a line is the **rate of change** of y with respect to x and can be calculated using the coordinates of two points on the line. Let (x_1, y_1) and (x_2, y_2) be the two points. Then $y_2 - y_1$ is the vertical change (the change in the dependent variable), and $x_2 - x_1$ is the horizontal change (the change in the independent variable). The slope, or rate of change, is the quotient of these changes.



$$\text{slope} = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}$$

$$= \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Definition of Slope

The **slope** of the line through the two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

Example

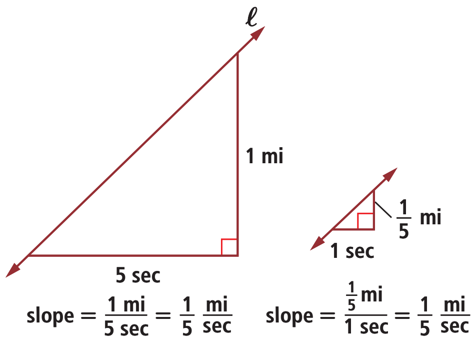
Determine the slope of the line with equation $d = \frac{1}{5}t$, where t is the independent variable time (in seconds) and d is the dependent variable distance (in miles).

Solution Use the definition of slope. Because d is on the vertical axis and t is on the horizontal axis, the ordered pairs are of the form (t, d).

Find two points on the line; either point can be considered (t_1, d_1). Here we use $(t_1, d_1) = (10, 2)$ and $(t_2, d_2) = (15, 3)$.

$$\text{slope} = \frac{d_2 - d_1}{t_2 - t_1} = \frac{3 \text{ mi} - 2 \text{ mi}}{15 \text{ sec} - 10 \text{ sec}} = \frac{1}{5}\frac{\text{mi}}{\text{sec}}$$

Refer to the graph at the beginning of this lesson. Notice that for every change of 5 units to the right, there is a change of 1 unit up. This is equivalent to saying that for every change of 1 horizontal unit, there is a change of $\frac{1}{5}$ of a vertical unit. Notice that because of the difference in units, you cannot visually see the slope as $\frac{1}{5}$.



STOP QY2

> **QY2**
> What is the slope of the line through (9, 36) and (25, 100)?

The Slope of $y = kx$

Observe from the Example that the graph of $y = \frac{1}{5}x$ has slope $\frac{1}{5}$. This is an instance of the following theorem.

> **Slope of $y = kx$ Theorem**
>
> The graph of the direct-variation function with equation $y = kx$ has constant slope k.

Proof Let (x_1, y_1) and (x_2, y_2) be any two distinct points on $y = kx$, with $k \neq 0$. Since the points are on the line,

$y_1 = kx_1$ substitution

and $y_2 = kx_2$

Now solve this system of equations for k.

$y_2 - y_1 = kx_2 - kx_1$ Subtraction Property of Equality

$y_2 - y_1 = k(x_2 - x_1)$ Distributive Property

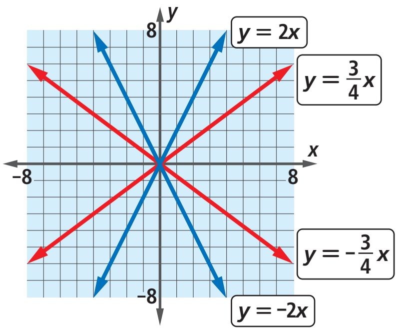
$\frac{y_2 - y_1}{x_2 - x_1} = k$ Division Property of Equality

So by the definition of slope, k is the slope of the line through these points.

Thus, k is the slope of the line with equation $y = kx$.

All the lines with equation $y = kx$, where k is any real number, make up a family of lines through the origin with different slopes k. They are all direct-variation functions. In general, the domain of a function with equation $y = kx$ is the set of real numbers, and the range is the set of real numbers.

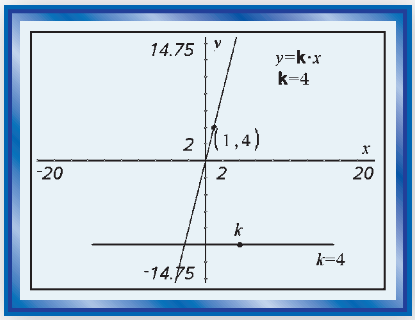
To explore how the value of k affects the graph of $y = kx$, you can graph $y = kx$ using several different values of k. This approach lets you view several graphs simultaneously and compare them. The graphs of $y = kx$ for four different values of k are shown on the axes below.



Activity

MATERIALS CAS, graphing calculator, or variation graph application

In this Activity, you will vary k to see how the graph of $y = kx$ changes.



Step 1 Consider the family of curves $y = kx$. Examine the graph for various positive values of k. Describe, in terms of slope, how the graph of the line $y = kx$ behaves for values of $k > 0$. How is the steepness of the graph affected as k increases?

Step 2 Examine the graph for various negative values of k. Describe, in terms of slope, how the graph of the line $y = kx$ behaves for values of $k < 0$. How is the steepness of the graph affected as k decreases?

Step 3 Describe the graph of the line $y = kx$ when $k = 0$.

Step 4 Choose any k and note the coordinates of the point $A = (1, k)$ on the graph. A always has x-coordinate 1, but its y-coordinate changes with the value of k. Use the coordinates of point A and the fact that the line passes through the origin to find the slope of the line.

Compare this slope to the y-coordinate of A and the value of k. What do you notice?

Questions

COVERING THE IDEAS

1. By definition, $\frac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line through which points?

2. Use the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for the slope of a line.
   a. Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (4.2, -5.3)$. Find the slope of the line through these two points.
   b. Let $(x_1, y_1) = (4.2, -5.3)$ and $(x_2, y_2) = (2, 4)$. Again, find the slope of the line through these points. Compare your answer to your answer to Part a.
   c. Mingmei incorrectly calculated the slope of the line through $(x_1, y_1) = (4.2, -5.3)$ and $(x_2, y_2) = (2, 4)$ as follows: $\frac{4 - (-5.3)}{4.2 - 2} \approx 4.23$. What error did she make? How does her answer compare to the answer you found in Part b?
   d. Given any two points $A = (c, d)$ and $B = (j, k)$, is the slope of the line through A and B the same as the slope of the line through B and A? Explain using the results you found in Parts a and b.

3. **Fill in the Blanks** Slope $= \frac{\text{change in the \_\_?\_\_ variable}}{\text{change in the \_\_?\_\_ variable}}$

4. **Fill in the Blanks** A slope of $-\frac{2}{5}$ means that for every change of 5 units to the right there is a change of __?__ units __?__. It also means that for every change of 1 horizontal unit there is a vertical change of __?__ unit.

5. In the graph of $d = \frac{1}{5}t$, the slope of the line is $\frac{1}{5}$. Write a sentence to describe this slope in the context of lightning and thunderstorms, using the appropriate units.

6. In the lesson, a triangle is drawn to show that a line has slope $\frac{1}{5}$. Draw a similar diagram that shows that a line has a slope of $\frac{7}{3}$.

7. **Fill in the Blanks** The graph of $y = kx$ slants up as you read from left to right if k is __?__. It slants down as you read from left to right if k is __?__.

8. When k is negative, in which quadrants is the graph of $y = kx$?

9. **Fill in the Blanks** The graph of every direct-variation function $y = kx$ is a __?__ with slope __?__ and passing through the point __?__.

APPLYING THE MATHEMATICS

In 10–12, compute the rate of change for the given situation. Include units where appropriate.

10. An escalator drops 2 feet for every 7 seconds traveled.

11. A car travels forward 60 miles every hour.

12. A line passes through the points $\left(-2, \frac{1}{3}\right)$ and $\left(\frac{2}{5}, 10\right)$.

13. Refer to Step 4 of the Activity. Find an additional point B on the line $y = kx$ for your chosen value of k. Calculate the slope of the line using points A and B. Compare this to the slope you calculated using A and O. How do the slopes compare to each other and to k?

14. Bicycle manufacturers have found that, for the average person and a given style of bicycle, the proper seat height h varies directly with the inseam measurement i of the rider's pants.

a. If the seat height is 27 inches for an inseam of 25 inches, determine an equation for the variation function.

b. Should the domain and range be all real numbers? Why or why not?

c. What is the slope of the graph of this function?

d. What does the slope represent in this situation?

15. Plot the points (3, 1), (9, 3), (12, 6) on a coordinate plane. Do these points lie on a line that represents a direct-variation function of the form $y = kx$? If so, compute the rate of change of that function. If not, explain why not.

16. The federal minimum wage was \$5.15/hr between 1997 and the summer of 2007.

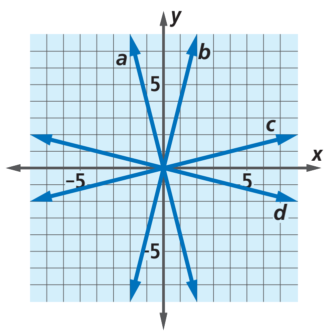
a. Determine the direct-variation function that calculates the amount of money m that someone earned (before taxes) by working w hours at this minimum wage.

b. A typical work week has 40 hours. Determine a reasonable domain and range for the function that maps time worked in a week onto the amount earned.

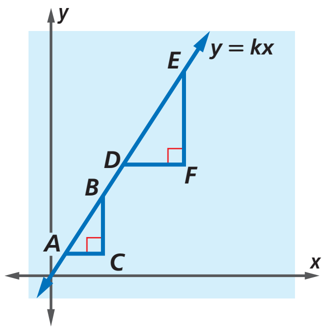
c. Sketch a detailed graph of the function.

d. In July 2009, the minimum wage increased to \$7.25 an hour. Describe how the graph of the function would change.

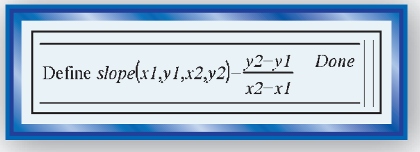
17. The graph at the right shows the four equations $y = 4x$, $y = -4x$, $y = \frac{1}{4}x$, and $y = -\frac{1}{4}x$. Match each graph with its equation.



18. Below is a graph of $y = kx$. Explain how similar triangles can be used to show that the slope of the line is the same no matter which points are chosen to find the slope.



19. To program the formula for slope into a CAS, use the `DEFINE` command and enter slope$(x1, y1, x2, y2) = \frac{y_2 - y_1}{x_2 - x_1}$. Then use your programmed formula to find the slope of the line through the given points.



a. (152, 278) and (194, 360)   b. (8.4, -3.6) and (6, 10)

REVIEW

20. In the variation function $W = \frac{k}{d^3}$, what is the effect on W if

a. d is tripled?

b. d is halved? **(Lesson 2-3)**

21. Assume that the cost of a square cake varies directly as the square of the length of a side. What would be the ratio of the cost of a 6-inch square cake to a 10-inch square cake? **(Lesson 2-3)**

In 22–25, state whether the equation is of a direct-variation function, an inverse-variation function, or neither. (Lessons 2-1, 2-2)

22. $y = -\frac{3}{x}$ 23. $y = -\frac{x}{3}$

24. $y = -\frac{3}{x^2}$ 25. $y = x - 3$

26. Ohm's Law, $I = \frac{V}{R}$, relates current I (in amperes) to voltage V (in volts) and resistance R (in ohms). **(Lesson 1-7)**

a. Solve this formula for R.   b. Solve this formula for V.

EXPLORATION

27. Each of the following terms is a synonym for *slope*. Research each term to find out who might use each one.

a. marginal cost   b. pitch

c. grade   d. rise over run

QY ANSWERS

1. In Quadrants II, III, and IV, t or d or both would be negative. The distance d cannot be negative, and the thunder cannot occur before the lightning, so the time t cannot be negative.

2. $\frac{100 - 36}{25 - 9} = \frac{64}{16} = 4$