Lesson **2-3**

The Fundamental Theorem of Variation

BIG IDEA When y varies directly as x^n, multiplying x by c causes y to be multiplied by c^n; when y varies inversely as x^n, multiplying x by c causes y to be divided by c^n.

In previous lessons, you explored the effects of doubling or tripling the length s of the edge of a cube on the surface area S.A. of the cube. You also explored the effects of doubling or tripling the value of w on the value of t in the equation $t = \frac{48}{w}$. In this lesson you will see how to generalize the findings of these problems.

This Activity explores how changes in the independent variable result in changes in the dependent variable in two different variation functions.

Activity

Fruit Roll Industries makes mini and regular fruit pops and is considering making larger jumbo pops. Assume the production cost p of a fruit pop varies directly with the volume V of candy on the stick. So, $p = k_0 \cdot V$. Also assume that a pop is approximately a sphere, so $V = \frac{4}{3}\pi r^3$, where r is the mean radius of pops of a particular size. Then $p = k_0 \cdot \frac{4}{3}\pi r^3$. Since $\frac{4}{3}$, π, and k_0 are all constants, $\frac{4}{3}\pi k_0$ is a constant and we can write $p = kr^3$. This means that p varies directly with the cube of r.

Step 1 Make a table like the one below based on the estimated radius of different-size pops. Fill in the blank cells in all columns except the one labeled Production Cost Estimate.

Size	Mean Radius r (cm)	r^3 (cm^3)	Ratio of Radius to Mini's Radius	Ratio of r^3 to Mini's r^3	Production Cost Estimate
Mini	0.7	0.343	1:1	1:1	$\frac{1}{2}$ cent
Regular	1.4	?	?	?	?
Jumbo	2.1	?	?	?	?

Mental Math

Claudia is building an in-ground pool in her back yard. The pool is 12 feet wide and 20 feet long.

a. If the pool is 5 feet deep, how much water can it hold?

b. Claudia is building a cement walkway 2 feet wide around the pool. If she builds a fence around the walkway, how long will the fence be?

c. If she places a fence post every 4 feet around the walkway, how many posts will she need?

National Lollipop Day is celebrated on July 20.

Step 2 Compare r and r^3 for the mini and regular pops.

a. The radius of the regular pop is __?__ times as big as the mini pop, but r^3 for the regular pop is __?__ times as big as for the mini pop. Since p varies directly as r^3, p is proportional to r^3. So, the cost of production of a regular pop should be __?__ times that of the mini, pop, or __?__ cents.

b. Make the same comparisons for the mini and jumbo pops. The jumbo cost should be __?__ times the mini cost, or __?__ cents. Note that when the mini's radius is multiplied by 3 to get the jumbo radius, you can find the cost of the jumbo pop by multiplying the cost of the mini pop by __?__ to the third power.

c. Suppose there is a super jumbo pop with a radius of 2.8 cm. Fill in the table to find the estimated cost of production? Justify your answer.

Size	Mean Radius r (cm)	r^3 (cm^3)	Ratio of Radius to Mini's Radius	Ratio of r^3 to Mini's r^3	Production Cost Estimate
Super Jumbo	2.8	?	?	?	?

Step 3 Complete the following sentence to generalize your findings: If the production cost varies directly as r^3, and r is multiplied by a number c, then the cost is multiplied by __?__.

Step 4 Suppose there is only one size of carton used for fruit pop shipments and each carton contains fruit pops of only one size. As the radius of the fruit pop gets larger, what happens to the number of pops that fit in the carton? The number n of pops that fit in a carton varies ____?____ as the cube of the radius. Algebraically, $n = \frac{k}{r^3}$.

a. Make a table like the one below and fill in the ratio column.

Size	Radius r (cm)	Ratio of r^3 to Mini's r^3	Number of Pops n in a Carton
Mini	0.7	1:1	270
Regular	1.4	?	?
Jumbo	2.1	?	?
Super Jumbo	2.8	?	?

b. Note that since n varies inversely as r^3, n will ____?____ as r^3 increases. Since r^3 for the regular pop is __?__ r^3 for the mini pop, $\frac{270}{?}$ = __?__ regular pops will fit in a carton. Write this number in the table.

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c. Note that when the mini's radius is multiplied by 3 to get the jumbo's radius, you can find the number of jumbo pops that fit in the carton by ___?___ the number of mini pops by ___?___ to the ___?___ power. You can find the number of super jumbo pops by ___?___ the number of mini pops by ___?___ to the ___?___ power. Use this information to complete the rightmost column of the table.

Step 5 Complete the following sentence to generalize your findings: If the number of pops in a carton varies inversely as r^3, and r is multiplied by a number c, then the number of pops is divided by ___?___.

STOP **QY1**

> **QY1**
> If an all-day giant pop has a radius that is 5 times the radius of the mini pop, what is its production cost?

The generalizations you made in Steps 3 and 5 of the Activity are instances of the *Fundamental Theorem of Variation*. In your generalizations, $n = 3$. In the theorem, n can be any positive number.

The Fundamental Theorem of Variation

1. If $y = kx^n$, that is, y varies *directly* as x^n, and x is multiplied by c, then y is multiplied by c^n.
2. If $y = \frac{k}{x^n}$, that is, y varies *inversely* as x^n, and x is multiplied by a non-zero constant c, then y is divided by c^n.

Proof of 1 Let y_1 = original value before multiplying x by c

$y_1 = kx^n$ — definition of direct variation

Let y_2 = value when x is multiplied by c

To find y_2, x must be multiplied by c.

$y_2 = k(cx)^n$ — definition of y_2

$y_2 = k(c^n x^n)$ — Power of a Product Postulate

$y_2 = c^n(kx^n)$ — Associative and Commutative Properties of Multiplication

$y_2 = c^n y_1$ — substitution of y_1, for kx^n

Proof of 2 The proof of this part is left for you to do in Question 17.

STOP **QY2**

> **QY2**
> If y varies *inversely* as x, and x is divided by c, what is the effect on y?

Example 1

Fruit Roll Industries needs a cost estimate for the wrappers of the new super jumbo pop. Because the pop is roughly a sphere, the area of the wrapper is a multiple of the surface area of a sphere, where $S.A. = 4\pi r^2$. So, the cost w of a wrapper varies directly with the surface area. The company knows the wrappers for the mini pops cost 1.5¢ each. How much do the wrappers of the super jumbo pop cost?

Solution Because the cost w_1 of a mini pop wrapper varies directly with the surface area, $w_1 = kr^2$ when r is the radius of the mini pop. In the Activity, the radius of the super jumbo pop was 4 times as long as the radius of the mini pop, or $4r$. So, if w_2 is the cost of a super jumbo pop wrapper, $w_2 = k(4r)^2$. Then

$w_2 = k(4^2r^2)$ Power of a Product Property

$w_2 = 16(kr^2)$ Associative and Commutative Properties of Multiplication

$w_2 = 16w_1$ Substitution of w_1 for kr^2

So, the cost of a super jumbo pop wrapper is 16, or 4^2, times the cost of a mini pop wrapper. Because a mini pop wrapper costs 1.5¢, a super jumbo pop wrapper costs 16(1.5) = 24¢.

GUIDED

Example 2

Generally, the weight of a land animal of a particular type varies directly with the cube of its femur diameter. Phoberomys, an extinct rodent that lived over 5 million years ago, is an ancestor of the modern guinea pig. Its femur diameter is 18 times that of today's average guinea pig. Estimate how the weight of this ancient rodent compares to the weight of a modern guinea pig.

Solution 1 Let d = the animal's femur diameter and w = the animal's weight. Since weight varies directly as the cube of femur diameter, an equation for the variation function is $w = kd^3$. Now apply the Fundamental Theorem of Variation. When d is multiplied by __?__, w is multiplied by $\underline{\ ?\ }^3 = \underline{\ ?\ }$. Thus, the ancient rodent Phoberomys weighed about __?__ times as much as a modern guinea pig.

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This is what a Phoberomys may have looked like.

Illustration by Carin L. Cain © Science

Solution 2 Set the problem up as in Example 1.

An equation for the variation function is $w = kd^3$. Let w_1 = the weight of the guinea pig, d = the femur diameter of the guinea pig, and w_2 = the weight of Phoberomys with femur diameter 18d.

$$\begin{aligned} w_1 &= kd^3 \\ w_2 &= k(\underline{\ ?\ })^3 \\ &= k(\underline{\ ?\ }d^3) \\ &= 18^3(\underline{\ ?\ }) \\ &= 18^3 w_1 \\ &= \underline{\ ?\ }w_1 \end{aligned}$$

So, $w_2 \approx 5800w_1$. Phoberomys weighed about ___?___ times as much as the modern guinea pig.

A modern adult male guinea pig weighs about 2 pounds. So, with this model, an estimate of Phoberomys' weight is 11,600 pounds, or almost 6 tons! Other measurements of Phoberomys show that the $w = kd^3$ model overestimates the weight of the rodent. A better model is $w = kd^{2.5}$. You will study noninteger exponents in a later chapter.

Questions

COVERING THE IDEAS

In 1 and 2, refer to the Activity.

1. What is the production-cost estimate of a jumbo pop?
2. How many super jumbo pops fit in a carton?

In 3 and 4, consider the formula $S.A. = 4\pi r^2$ for the surface area of a sphere from Example 1.

3. **Fill in the Blank** The pairs $(2, 16\pi)$ and $(4, 64\pi)$ represent (r, s) for two spheres. They are instances of this pattern: if the radius r is doubled, the surface area is multiplied by ___?___.
4. Show that the Fundamental Theorem of Variation is true for the points in Question 3.
5. **Fill in the Blank** If $y = kx^n$, and x is divided by c, then y is ___?___.
6. **Fill in the Blank** If $y = \frac{k}{x^n}$, and x is multiplied by c $(c \neq 0)$, then y is ___?___.

7. Refer to Example 2. Suppose the diameter of the femur bone of another ancient rodent is 3.2 times that of the modern guinea pig. Compare the weight of this ancient rodent to the weight of a modern one.

In 8–10, suppose $y = 3x^4$.

8. Complete the table of values at the right.

x	y
1	?
2	?
3	?
4	?
6	?
8	?
9	?

9. Describe the change in y when x is doubled. Explain your reasoning.
10. When answering the question, "Describe the change in y when x is divided by 3," Laura's response was "y is multiplied by $\frac{1}{81}$." Do you agree with Laura? Why or why not?
11. The volume of a cube varies directly with the cube of its edge length.
    a. If the edge length is multiplied by 8, what effect does that have on the volume?
    b. If the edge length is divided in half, what effect does that have on the volume?

APPLYING THE MATHEMATICS

12. Marta went to the farmer's market to buy oranges. The oranges that are 3 inches in diameter cost 25 cents per dozen. The oranges that are 4 inches in diameter are 50 cents per dozen. Marta chose the 3-inch oranges. Did she make the more economical decision? Explain your answer?
13. The Brobdingnagians in Jonathan Swift's *Gulliver's Travels* are similar to us, but they are 12 times as tall.
    a. How would you expect the weight of these giants to compare to our weight?
    b. How would you expect their surface area to compare to ours?
14. The inverse square law for light intensity, $I = \frac{k}{d^2}$, models the relationship between distance d from a light source and the intensity I of the light.
    a. When Booker is working at his computer, he is twice the distance from the floor lamp in his room as he is when he is working at his desk. Compare the light intensity of the floor lamp at his computer to the intensity at his desk. Justify your answer.
    b. Suppose Booker moves his computer so that it is three times as far from the floor lamp as his desk. Compare the intensity of the lamp light at his computer and his desk. Justify your answer.

In 15 and 16, refer to the illustration of the micro- and standard-size CDs at the right. The radius of the standard CD is 5 inches and the radius of the micro CD is 3 inches.

A standard and micro CD.

15. What is the ratio of the circumferences of the larger to the smaller CD?
16. Use the Fundamental Theorem of Variation to calculate the ratio of the surface areas of the larger to the smaller CD, assuming the CDs had no hole in the center.
17. Complete the proof of Part 2 of the Fundamental Theorem of Variation.

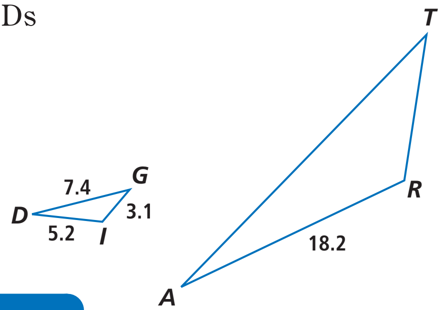
REVIEW

18. Translate this statement into a variation equation: The number of oranges that fit into a crate is inversely proportional to the cube of the radius of each orange. **(Lesson 2-2)**
19. **Multiple Choice** Most of the power of a boat motor results in the generation of a wake (the track left in the water). The engine power P used in generating the wake is directly proportional to the seventh power of the boat's speed s. Which equation models this variation? **(Lesson 2-1)**

    A $P = 7s$ B $s = kP^7$ C $P = ks^7$ D $P = k^7s$
20. Suppose V varies directly as the third power of r. If $V = 32$ when $r = 8$, find V when $r = 5$. **(Lesson 2-1)**
21. a. If one blank CD costs c cents, how much do n blank CDs cost? **(Lesson 1-1)**

    b. If two blank CDs cost d cents, how much do m blank CDs cost? **(Lesson 1-1)**
22. ***Skill Sequence*** Find all solutions. **(Previous Course)**

    a. $x^2 = 25$ b. $25y^2 = 36$ c. $3z = \frac{25}{3z}$
23. In the diagram at the right, $\triangle DIG \sim \triangle ART$. Find TR. **(Previous Course)**



EXPLORATION

24. Go to the store or use the Internet to find a product that comes in two different sizes (for example, a regular and a large-screen television). Compare prices of the smaller and larger versions of the product. Using the Fundamental Theorem of Variation, decide whether the prices of the two products are in the proper ratio. Explain your decision. What factors other than size might affect the price?

QY ANSWERS

1. 62.5 cents
2. y is multiplied by c.