Lesson 2-1 Direct Variation

Vocabulary

varies directly as

direct-variation equation

constant of variation

direct-variation function

directly proportional to

BIG IDEA When two variables x and y satisfy the equation $y = kx^n$ for some constant value of k, we say that y varies directly as x^n.

Mental Math

a. Which is larger: $\frac{7}{9}$ or $\frac{7}{10}$?

b. Which is closer to $\frac{1}{2}$: $\frac{3}{5}$ or $\frac{4}{7}$?

c. Which is closer to zero: $\frac{1}{2} - \frac{1}{3}$ or $\frac{1}{3} - \frac{1}{4}$?

d. Which is closer to 1: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$ or $\frac{1}{3} + \frac{1}{5} + \frac{1}{7}$

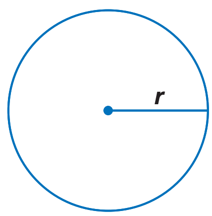
When possible, Lance puts his bike into its highest gear for maximum speed. In highest gear, the ratio of the number of front-gear teeth to the number of back-gear teeth is $\frac{52 \text{ teeth}}{11 \text{ teeth}} \approx 4.73$. This means that as Lance turns the pedals one complete revolution, the back wheel turns almost 5 times. So if w is the number of back-wheel turns per minute and p is the number of pedal turns per minute, then

$$w = \frac{52}{11}p.$$

If Lance starts pedaling twice as fast, then the back wheel will also turn twice as fast as it previously did. We say that w **varies directly as** p, and we call $w = \frac{52}{11}p$ a **direct-variation equation**. In a direct variation, both quantities increase or decrease together.

Suppose Lance changes to a lower gear with a gear ratio of $\frac{42 \text{ teeth}}{21 \text{ teeth}} = 2$. A direct-variation equation for this situation is $w = 2p$.

Recall the formula $A = \pi r^2$ for the area of a circle. This, too, is a direct-variation equation; as the radius r increases, the area A also increases. In this case, A varies directly as r^2. Often this wording is used: the area A varies directly as the *square* of r.



$A = \pi r^2$

Direct-Variation Functions

The formulas $w = \frac{52}{11}p$, $w = 2p$, and $A = \pi r^2$ are all of the form $y = kx^n$, where k is a nonzero constant, called the **constant of variation**, and n is a positive number. These formulas all describe *direct-variation functions*.

Definition of Direct-Variation Function

A **direct-variation function** is a function that can be described by a formula of the form $y = kx^n$, with $k \neq 0$ and $n > 0$.

When y varies directly as x^n we also say that y is **directly proportional to** x^n. For instance, the formula $A = \pi r^2$ can be read "the area of a circle is directly proportional to the square of its radius." Here $n = 2$ and $k = \pi$, so π is the constant of variation.

STOP **QY1**

If you know one point (x, y) of a direct-variation function, you can determine k and, thus, know precisely the function.

The formula $A = \pi r^2$ is one example of a general theorem from geometry: in a set of similar figures (in this case, circles), area is proportional to the square of length. Likewise, in a set of similar figures, volume is proportional to the cube of edge length. Example 1 is an instance of that theorem.

QY1

Identify the constant of variation and the value of n in the formula $w = \frac{52}{11}p$.

Example 1

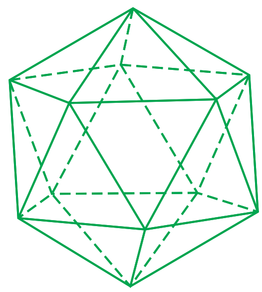
The volume V of a regular icosahedron is directly proportional to the cube of the length ℓ of an edge.

a. Identify the dependent and independent variables and write an equation relating them.

b. A regular icosahedron with an edge length of 4 cm has a volume of about 140 cm^3. Determine the constant of variation k.

c. Rewrite the variation equation using your result from Part b.

d. Approximate to the nearest cubic centimeter the volume of an icosahedron with an edge length of 5 cm.



Regular Icosahedron (20 faces)

Solution

a. Because V is directly proportional to ℓ, the dependent variable is V and the independent variable is ℓ. In this problem $n = 3$, so an equation for the direct variation is $V = k\ell^3$, where k is a constant.

b. To determine k, substitute $V = 140$ cm^3 and $\ell = 4$ cm into your direct-variation equation from Part a.

$$140 \text{ cm}^3 = k \cdot (4 \text{ cm})^3$$

$$140 \text{ cm}^3 = (64 \text{ cm}^3) \cdot k$$

$$k = \frac{140 \text{ cm}^3}{64 \text{ cm}^3} = 2.1875$$

(continued on next page)

c. Substitute $k = 2.1875$ to get a formula relating the edge length and volume.

$$V = 2.1875\,\ell^3$$

d. Evaluate your formula when $\ell = 5$ cm.

$$V = 2.1875(5\text{ cm})^3 \approx 270\text{ cm}^3$$

Direct-Variation Functions

The four parts of Example 1 illustrate a procedure you can use to solve variation problems. First, write a general equation that describes the variation. Next, substitute the given values into the general equation and solve for k. Then use the k-value to write the variation function. Finally, evaluate the function at the specified point to find the missing value.

Guided Example 2 illustrates how to use this procedure to solve a typical direct-variation problem.

GUIDED

Example 2

Suppose b varies directly as the sixth power of g. If $b = 729$ when $g = 2$, find b when $g = 10$.

Solution Write an equation with variables b, g, and k that describes the variation.

$$\underline{\ ?\ } = k \cdot \underline{\ ?\ }^6$$

Find the constant of variation. You are given that $b = 729$ when $g = 2$. Substitute these values into the variation formula to find k.

$$\underline{\ ?\ } = k \cdot \underline{\ ?\ }^6$$

$$\underline{\ ?\ } = k \cdot \underline{\ ?\ }$$

$$\underline{\ ?\ } = k$$

Now rewrite the variation formula using the value of the constant you found.

$$\underline{\ ?\ } = \underline{\ ?\ } \cdot \underline{\ ?\ }^6$$

Finally, use the formula to find b when $g = 10$.

$$b = \underline{\ ?\ } \cdot \underline{\ ?\ }^6$$

$$b = \underline{\ ?\ }$$

So, when $g = 10$, $b = \underline{\ ?\ }$.

Activity

You can solve direct-variation problems on a CAS by defining two functions. The first function calculates the constant of variation *k*. The second function calculates values of the direct-variation function using that *k* and a value of the independent variable.

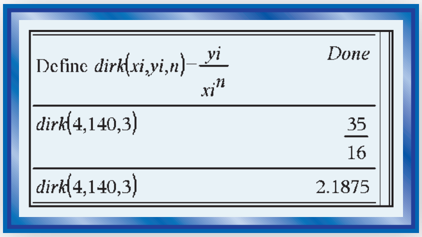
Step 1 Plan your first function.

a. Give a meaningful name to the function so you can use it in the future. We call the first function *dirk*, short for *direct-variation k-value*.

b. Think about the inputs you need to calculate k and give them names. In Part b of Example 1, you used initial values of the independent and dependent variables and the value of the exponent to calculate k. Good names for the initial values of the independent and dependent variables are *xi* and *yi*, respectively. We call the exponent n, as in the direct-variation formula $y = kx^n$.

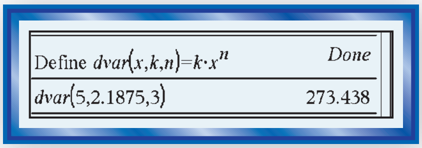
c. Generalize the result of solving for k in Part b of Example 1.
In the Example,
$xi = 4$, $yi = 140$, $n = 3$, and $k = \frac{140}{4^3}$.
So, in general, $k = \frac{yi}{xi^n}$.

Step 2 Clear the values for *xi, yi, k,* and n in the CAS memory. Then define the *dirk* function using its three inputs and the general formula for k you found above.



Step 3 Use the *dirk* function to calculate k for the situation in Example 1. The input values are $xi = 4$, $yi = 140$, and $n = 3$. The display at the right shows the value of k in both fraction and decimal form.

Step 4 A good name for the second function is *dvar*, for *direct variation*. There are three inputs for this function also: another known value x for the independent variable, the constant of variation k, and the exponent n. Clear x from the CAS memory and define *dvar* using the general form of the direct-variation formula, as shown at the right.



Step 5 Use *dvar* to find the missing function value for $x = 5$, $k = 2.1875$, and $n = 3$. Compare your answer to Part d of Example 1.

STOP **QY2**

You may use *dirk* and *dvar* to answer the Questions.

▶ QY2

Use *dirk* and *dvar* to check your answers to Example 2.

Questions

COVERING THE IDEAS

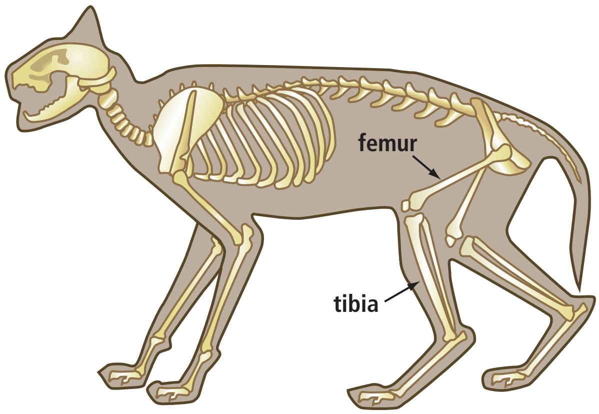
1. Provide an example of direct variation from everyday life.

2. Give an example of a direct-variation function from geometry.

3. **Fill in the Blanks** In the function $y = 5x^2$, __?__ varies directly as __?__, and __?__ is the constant of variation.

4. Suppose pay varies directly as time worked and Paul makes $400 for working 25 hours.
   a. What is the constant of variation?
   b. How much will Paul make for working 32 hours?

5. The lengths of the femur and tibia within a species of mammal are typically directly proportional. If the femur of one household cat is 116.0 mm long and its tibia is 122.8 mm long, how long is the tibia of a cat whose femur is 111.5 mm long?



6. Suppose $s = 4.7t$.
   a. Find s when $t = 3.1$.
   b. Is this an equation for a direct-variation function? How can you tell?

7. Assume that y is directly proportional to the square of x.
   a. **Multiple Choice** Which equation represents this situation?
      A $y = 2x$ B $y = kx^2$ C $x = ky^2$ D $y = 2x^k$
   b. **Multiple Choice** It is also true that x is directly proportional to
      A the square of y. B the square root of y.
      C twice y. D half of y.

In 8 and 9, refer to Example 1.

8. What is the volume of a regular icosahedron with an edge of length 2 cm?

9. Use the `solve` command on a CAS to find the length of the edge of a regular icosahedron with a volume of 20 cm^3.

10. Suppose W varies directly as the fourth power of z, and $W = 27$ when $z = 3$.
    a. Find the constant of variation.
    b. Find W when $z = 9$.

11. Suppose y varies directly as the cube of x, and $y = 27.6$ when $x = 0.5$. Find y when $x = 3.2$.

12. Suppose f is a function defined by $f(x) = \frac{x^3}{7}$. Is f a direct-variation function? Justify your answer.

APPLYING THE MATHEMATICS

13. When lightning strikes in the distance, you do not see the flash and hear the thunder at the same time. You first see the lightning, then you hear the thunder.
    a. Write an equation to represent the following situation: The distance d (in miles) from the observer to the flash varies directly as the time t (in seconds) between seeing the lightning and hearing the thunder.
    b. Suppose that lightning strikes a point 4 miles away and that you hear the thunder 20 seconds later. How far away has lightning struck if 12 seconds pass between seeing a flash and hearing its thunder?

14. The speed of sound in air is about 1,088 feet per second.
    a. Convert the speed of sound to miles per second.
    b. Write the relationship between the number of miles sound travels and the number of seconds it takes sound to travel that distance as a direct variation. Be sure to identify what your variables represent.
    c. Use your answer from Part b to find the time it takes sound to travel four miles in air.

15. Refer to the formula S.A. $= 6s^2$ for the surface area of a cube with an edge of length s.
    a. Complete the table at the right.
    b. How many times as large is the area when $s = 4$ as when $s = 2$?
    c. How many times as large is the area when $s = 6$ as when $s = 3$?
    d. How many times as large is the area when $s = 10$ as when $s = 5$?
    e. **Fill in the Blank** Relate the length of the edge to the surface area of the cube. When the edge length doubles, the surface area ___?___.
    f. **Fill in the Blank** When the edge length triples, the surface area ___?___.

s	S.A.
1	?
2	?
3	?
4	?
5	?
6	?
7	?
8	?
9	?
10	?

16. The table at the right gives a typical thinking distance, braking distance, and overall stopping distance for a car traveling at various speeds.

a. Show that the thinking distance t is directly proportional to the speed s by finding the constant of variation and writing a direct-variation formula.

b. Show that the braking distance b is directly proportional to the square of the speed s by finding the constant of variation and writing a direct-variation formula.

c. Is the overall stopping distance d directly proportional to the speed? Explain how you know.

Speed	Thinking Distance	Braking Distance	Overall Stopping Distance
20 mph	20 ft	20 ft	40 ft
30 mph	30 ft	45 ft	75 ft
40 mph	40 ft	80 ft	120 ft
50 mph	50 ft	125 ft	175 ft
60 mph	60 ft	180 ft	240 ft
70 mph	70 ft	245 ft	315 ft

REVIEW

17. The distance d in feet a body falls in t seconds is given by the formula $d = \frac{1}{2}gt^2$, where $g = 32\ \frac{\text{ft}}{\text{sec}^2}$ is the constant of gravitational acceleration. **(Lesson 1-4)**

a. How far will a body fall in 7 seconds?

b. How long did a body fall if it traveled 83 feet?

18. Graph the function f with the equation $f(x) = \frac{1}{x^2}$ in a standard window. What is the domain of f? **(Lesson 1-4)**

19. In the table at the right, is y a function of x? Justify your answer. **(Lesson 1-2)**

x	y
–1	1
–1	–1
–4	2
0	0
–18	–4

20. A dime has a diameter of 17.91 mm. **(Lesson 1-1)**

a. You place 15 dimes in a row. What is the length of your row of dimes?

b. Write an expression to describe what happens to the length of your row of dimes when you remove x dimes.

21. Suppose $\triangle ABC \sim \triangle DEF$ (that is, the two triangles are similar).

a. Sketch a possible diagram of this situation.

b. What can you say about the ratios $\frac{AB}{BC}$ and $\frac{DE}{EF}$? **(Previous Course)**

In Questions 22 and 23, write the expression as a power of 5. (Previous Course)

22. $5^2 \cdot 5^3$

23. $(5 \cdot 5)^3$

EXPLORATION

24. Use the Internet to find some common gear ratios that are used for different speed settings on racing bikes. Write and solve a direct-variation problem involving one of these gear ratios.

QY ANSWERS

1. $\frac{52}{11}$ is the constant of variation, and $n = 1$.

2. *dirk* gives $k \approx 11.39$, and *dvar* shows that $b = 11{,}390{,}625$ when $g = 10$. So, the answers check.