Lesson 13-9

Binomial and Normal Distributions

Vocabulary

probability function, probability distribution

binomial probability distribution, binomial distribution

normal distribution

normal curve

standard normal curve

standardized scores

▶ **BIG IDEA** As the number of trials of a binomial experiment increases, the graphs of the probabilities of each event and the relative frequencies of each event approaches a distribution called a *normal distribution*.

Mental Math

Tell whether each graph is the graph of a function.

a. the image of $y = \sin x$ under R_{90}

b. the image of $y = \cos x$ under $r_{x\text{-axis}}$

c. the image of $x = 12$ under $T_{2,8}$

d. the image of $x = 12$ under R_3

A Binomial Distribution with Six Points

Let $P(n)$ = the probability of n heads in 5 tosses of a fair coin. Then the domain of P is {0, 1, 2, 3, 4, 5}. By the Binomial Probability Theorem, and because the probability of heads = the probability of tails = $\frac{1}{2}$,

$$P(n) = \left(\frac{1}{2}\right)^5\binom{5}{n} = \frac{1}{32}\binom{5}{n}.$$

GUIDED

Example 1

a. Copy and complete the table shown below.

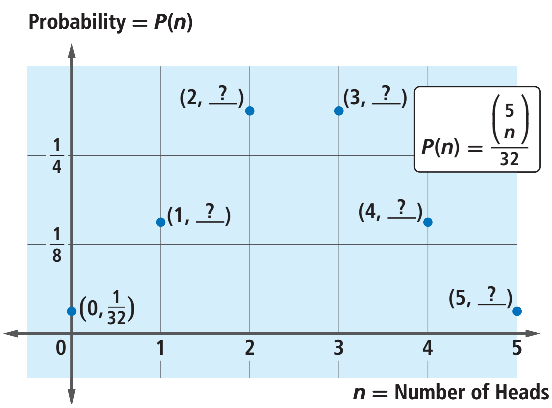
b. Graph and label the coordinates of the six points $(n, P(n))$.

Solution

a. Use the formula for $P(n)$ given above to fill in the second row.

n = Number of Heads	0	1	2	3	4	5
P(n) = Probability of n Heads in 5 Tosses of a Fair Coin	$\frac{1}{32}\binom{5}{0} = \frac{1}{32}$	$\frac{1}{32}\binom{5}{?} = \underline{\ ?\ }$	?	?	?	?

b. The points are graphed below. Fill in the missing coordinates with values from the table in Part a.



P is a *probability function*. A **probability function**, or **probability distribution**, is a function that maps a set of events onto their probabilities. Because the function P results from calculations of binomial probabilities, it is called a **binomial probability distribution,** or simply a **binomial distribution.**

A Binomial Distribution with Eleven Points

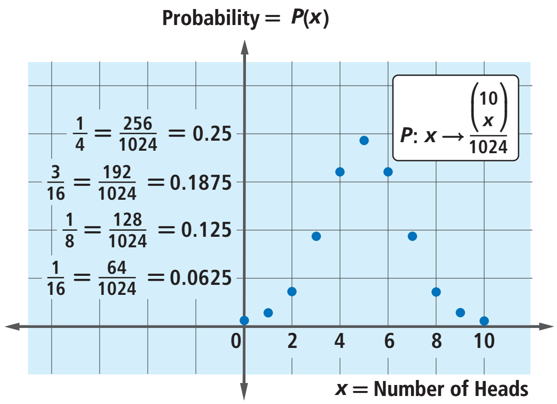
If a fair coin is tossed 10 times, the possible numbers of heads are 0, 1, 2, ..., 10, so there are 11 points in the graph of the corresponding probability function. Again, by the Binomial Probability Theorem with equally likely events, the probability $P(x)$ of tossing x heads is given by

$$P(x) = \left(\frac{1}{2}\right)^{10}\binom{10}{x} = \frac{1}{1024}\binom{10}{x}.$$

The 11 probabilities are easy to calculate because the numerators in the fractions are the numbers in the 10th row of Pascal's triangle. That is, they are binomial coefficients.

n = Number of Heads	0	1	2	3	4	5	6	7	8	9	10
$P(x)$ = Probability of x Heads	$\frac{1}{1024}$ ≈ 0.001	$\frac{10}{1024}$ ≈ 0.01	$\frac{45}{1024}$ ≈ 0.04	$\frac{120}{1024}$ ≈ 0.12	$\frac{210}{1024}$ ≈ 0.21	$\frac{252}{1024}$ ≈ 0.25	$\frac{210}{1024}$ ≈ 0.21	$\frac{120}{1024}$ ≈ 0.12	$\frac{45}{1024}$ ≈ 0.04	$\frac{10}{1024}$ ≈ 0.01	$\frac{1}{1024}$ ≈ 0.001

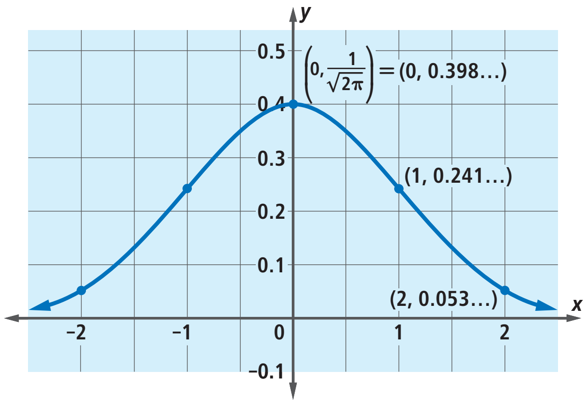
The binomial distribution in the table is graphed at the right. Closely examine this 11-point graph of $P(x) = \frac{1}{1024}\binom{10}{x}$, along with the table of values. The individual probabilities are all less than $\frac{1}{4}$. Notice how unlikely it is to get 0 heads or 10 heads in a row. (The probability for each is less than $\frac{1}{1000}$.) Even for 9 heads in 10 tosses, the probability is less than $\frac{1}{1000}$. Like the graph of the 6-point probability function $P(n) = \frac{1}{32}\binom{5}{n}$ on the previous page, this 11-point graph has a vertical line of symmetry.



Normal Distributions

As the number of tosses of a fair coin is increased, the points on the graph more closely outline a curve shaped like a bell. On the next page, this bell-shaped curve is positioned so that it is reflection-symmetric to the y-axis and its equation is simplest. Its equation is $y = \frac{1}{\sqrt{2\pi}}e^{\left(\frac{-x^2}{2}\right)}$.

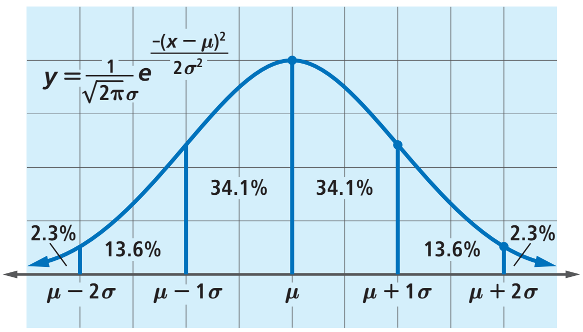
The function that determines this graph is called a **normal distribution**, and the curve is called a **normal curve**. Notice that its equation involves the famous constants $e \approx 2.718$ and $\pi \approx 3.14$. Every normal curve is the image of the graph at the right under a composite of translations and scale changes. Thus, the graph of $y = \frac{1}{\sqrt{2\pi}}e^{\left(\frac{-x^2}{2}\right)}$ is sometimes called the **standard normal curve**.



STOP QY1

In any distribution whose graph is a normal curve, the values fall in certain intervals based on the mean μ and standard deviation σ as shown in the graph below. For example, 34.1% of the function values fall between the mean and 1 standard deviation above the mean.

Normal Distribution Percentages



Normal curves are models for many natural phenomena. If you know the mean and standard deviation, you can determine other information about a normally distributed data set.

Example 2

The heights of men in the United States are approximately normally distributed with a mean of 69.2 inches and a standard deviation of 2.8 inches. What percent of men in the U.S. are taller than 72.0 inches?

Solution The difference between the mean height 69.2 inches and the given height 72.0 inches is 2.8 inches. This value is exactly one standard deviation for these data. So, 72.0 inches is 1 standard deviation above the mean, or $\mu + 1\sigma$. From the graph above, notice that to the right of $\mu + 1\sigma$ are 13.6% + 2.3% = 15.9% of the normally-distributed heights. So, about 15.9% of men are taller than 72.0 inches.

STOP QY2

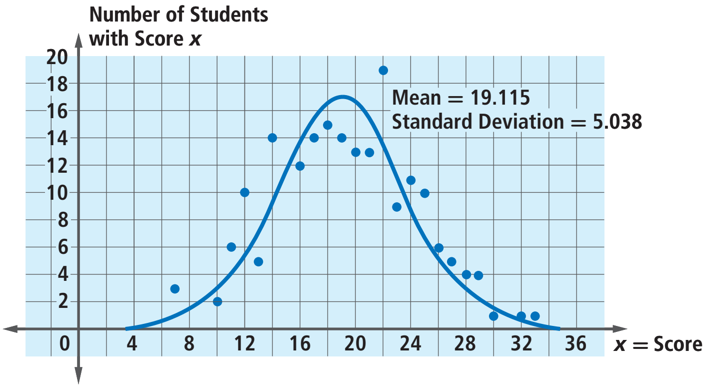
QY1

Find the *y*-values on the standard normal curve when $x = -1$ and $x = -2$.

QY2

It is estimated that 2.3% of the men in the U.S. are less than a certain height in inches. What is that height?

Normal curves are often good mathematical models for the distribution of scores on an exam. The graph at the right shows an actual distribution of scores on a 40-question test given to 209 geometry students. (It was a hard test!) A possible corresponding normal curve is shown.



On some tests, scores are **standardized**. This means that a person's score is not the number of correct answers, but is converted so that it lies in a normal distribution with a predetermined mean and standard deviation. Standardized tests make it easy to evaluate an individual score relative to the mean, but not to other individual scores.

Example 3

SAT scores are standardized to a historic mean of 500 and a standard deviation of 100. What percent of the scores are expected to be between 400 and 700?

Solution Find out how many standard deviations each score, 400 and 700, is from the mean, 500.

$$500 - 400 = 100 \text{ and } 700 - 500 = 200$$

So, a score of 400 is 1 standard deviation below the mean and a score of 700 is 2 standard deviations above the mean.

Refer to the Normal Distribution Percentages graph on the previous page. The percent of scores between $\mu - 1\sigma$ and $\mu + 2\sigma$ is $34.1\% + 34.1\% + 13.6\% = 81.8\%$.

About 81.8% of SAT scores are expected to be between 400 and 700.

The normal distribution is an appropriate topic with which to end this book, because it involves so many of the ideas you have studied in it. The distribution is a function. Its equation involves squares, square roots, π, e, and negative exponents. Its graph is the composite of a translation and scale change image of the curve with equation $y = \frac{1}{\sqrt{2\pi}}e^{\left(\frac{-x^2}{2}\right)}$. It models real data and shows a probability distribution that is used on tests that help to determine which colleges some people will attend. It shows how interrelated and important the ideas of mathematics are.

Questions

COVERING THE IDEAS

1. Let $P(n) = \frac{1}{32} \cdot \binom{5}{n}$.
   a. What kind of function is P?
   b. Find $P(4)$ and describe what it could represent.
2. What are the domain and range of the function $P: x \rightarrow \frac{1}{1024} \cdot \binom{10}{x}$?
3. If a fair coin is tossed 10 times, what is the probability of getting exactly 4 heads?
4. Let $P(n)$ = the probability of getting n heads in 7 tosses of a fair coin.
   a. Make a table of values for P.
   b. Graph P.
5. Write an equation for the standard normal curve.
6. Describe one application of normal curves.
7. a. What does it mean for scores to be standardized?
   b. What is one advantage of doing this?
8. Approximately what percent of scores on a normal curve are within 2 standard deviations of the mean?
9. Refer to Example 2. What percent of men in the U.S. are between 66.4 and 72.0 inches tall?
10. Approximately what percent of people score below 300 on an SAT test with mean 500 and standard deviation 100?

APPLYING THE MATHEMATICS

11. Graph the normal curve with mean 50 and standard deviation 10.
12. If you repeatedly toss 8 fair coins, about what percent of the time do you expect to get from 4 to 6 heads?
13. Some tests are standardized so that the mean is the grade level at which the test is taken and the standard deviation is 1 grade level. So, for students who take a test at the beginning of 10th grade, the mean is 10.0 and the standard deviation is 1.0.
    a. On such a test taken at the beginning of 10th grade, what percent of students are expected to score below 9.0 grade level?
    b. If a test is taken in the middle of 8th grade (grade level 8.5), what percent of students are expected to score between 7.5 and 9.5?
14. For the graduating class of 2004, total ACT scores for seniors ranged from 1 to 36 with a mean of 20.9 and a standard deviation of 4.8. What percent of students had an ACT score above 25?

15. **Fill in the Blanks** Assume that the lengths of jumps of boys on a track team are normally distributed with a mean of 18 feet and a standard deviation of 1 foot. In a normal distribution, 0.13% of the jumps lie more than 3 standard deviations away from the mean in each direction. This implies that about 1 out of __?__ boys will have a jump over __?__.

16. Let $y = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2}\right)}$. Estimate y to the nearest thousandth when $x = 1.5$.

17. A game has prizes distributed normally with a mean of \$60 and a standard deviation of \$20. If you play the game 10 times, what is the probability that you win more than \$100 at least 3 times?

REVIEW

18. In the Texas Lotto, a participant picks 6 numbers from 1 to 54 to win. What is the probability of *not* winning the Texas Lotto? **(Lesson 13-8)**

19. **True or False** The probability of getting exactly 50 heads in 100 tosses of a fair coin is less than 5%. Justify your answer. **(Lesson 13-7)**

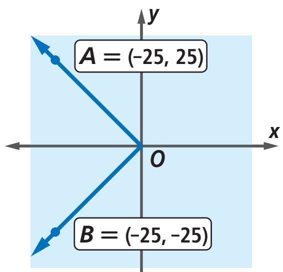
20. Evaluate ${}_nC_0$. **(Lesson 13-6)**

21. Expand $\left(6 - \frac{x}{2}\right)^5$. **(Lesson 13-6)**

22. A hot air balloon is sighted from two points on level ground at the same elevation on opposite sides of the balloon. From point P the angle of elevation is 21°. From point Q the angle of elevation is 15°. If points P and Q are 10.2 kilometers apart, how high is the balloon? **(Lessons 10-7, 10-1)**

23. Simplify $\sqrt{16} \cdot \sqrt{25} + \sqrt{-16} \cdot \sqrt{25} + \sqrt{-16} \cdot \sqrt{-25} + \sqrt{16} \cdot \sqrt{-25}$. **(Lessons 6-8, 6-2)**

24. Give an equation for the right angle AOB graphed at the right. **(Lesson 6-2)**



EXPLORATION

25. Together with some other students or using a random number generator, simulate the tossing of 12 coins and count the number of heads. Run the simulation at least 200 times. Let $P(h)$ = the number of times h heads appear out of 12.
    a. Graph the points $(h, P(h))$. How close is $P(h)$ to a normal distribution?
    b. What is the mean number of heads of the distribution?
    c. Estimate the standard deviation of the distribution.

QY ANSWERS

1. ≈ 0.24; ≈ 0.05
2. 2.3% of men are two standard deviations below the mean, and because one standard deviation equals 2.8 in., then the height is $69.2 - 5.6 = 63.6$ in.