Lesson

13-7 Probability and Combinations

Vocabulary

trial

binomial experiment

BIG IDEA The probability of an event occurring r times in n *trials* of a *binomial experiment* can be found by calculating combinations.

Mental Math

Calculate

a. ${}_{1000}C_{999}$

b. ${}_{1000}C_{1000}$

c. ${}_{1000}C_{1}$

d. ${}_{1000}C_{2}$

Pascal originally conceived of the triangle named in his honor in the context of probability problems. Activity 1 can help you connect Pascal's Triangle and probabilities.

Activity 1

MATERIALS penny or other coin

The task is to estimate the probability of getting exactly 2 heads in 4 tosses of the coin by repeating an experiment a large number of times.

Step 1 Make a table with headings as shown at the right. Include ten rows for trials 1 through 10. Flip a coin 4 times and record the results. For example, if the coin came up heads, then tails, then tails, and then heads, write HTTH in the Sequence column and 2 in the Number of Heads column.

Trial	Sequence	Number of Heads
1	?	?
2	?	?
⋮	?	?

Step 2 Repeat Step 1 nine more times until you have filled the entire table. Then tally the number of times you got 0 heads, 1 head, 2 heads, 3 heads, and 4 heads.

Step 3 Combine your results with others in your class. Then compute relative frequencies of each number of heads for the class as a whole. Graph the five ordered pairs, where each ordered pair is of this form: (number of heads in 4 flips, relative frequency of that number of heads). What is your class's relative frequency for getting 2 heads in 4 flips?

Many people are surprised to find out that the relative frequency of getting 2 heads in 4 flips is usually not too close to 50%. But it is not difficult to compute the probability if the coins are fair. Activity 2 explores that computation.

Activity 2

Step 1 Write every possible sequence of four H's and T's. Organize the sequences in a table like the one at the right. Three sequences have been written in the table to get you started. Then count the number of sequences in each cell and write the counts in the bottom row of the table.

n	0	1	2	3	4
Sequences of 4 Tosses with n Heads	?	THTT HTTT ?	HTTH ?	?	?
y = Number of Sequences with n Heads	?	?	?	?	?

Step 2 Graph the five points (n, y), where y is the number of sequences of 4 H's and T's that have n heads.

Step 3 Compare the shape of your graph in Step 2 with the shape of the graph in Step 3 of Activity 1.

You should find that the shapes of the graphs in the Activities are quite similar. In Activity 2 you should have found 6 different sequences of 2 heads and 2 tails:

HHTT, HTHT, HTTH, THHT, THTH, TTHH.

If the coin is fair, for a single toss, the probability $P(H)$ of heads and the probability of $P(T)$ tails each equal $\frac{1}{2}$. So, each of these 6 sequences has the same probability, $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$, regardless of the order of the heads and tails. Therefore, the probability of getting 2 heads in 4 tosses of a fair coin is $6 \cdot \frac{1}{16} = \frac{3}{8} = 0.375$. This should be close to your class's relative frequency from Step 3 of Activity 1.

You can use the same idea to find the probability of obtaining any number of heads in any number of tosses of a fair coin.

Example 1

Suppose a fair coin is flipped 6 times. What is the probability of obtaining exactly 2 heads?

Solution 1 First count the number of sequences of 6 flips with exactly 2 heads. You could list the sequences by hand, but the Combination Counting Formula gives a faster way to count. Number the six flips 1, 2, 3, 4, 5, and 6. Then choosing two flips to be heads is equivalent to choosing a two-element subset of {1, 2, 3, 4, 5, 6}. The number of two-element subsets of a six-element set is

$$_6C_2 = \binom{6}{2} = \frac{6!}{2!(6-2)!} = 15.$$

Now compute the probability of each sequence occurring. (Remember, each sequence has the same probability.)

$$P(H) \cdot P(H) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(T)$$
$$= P(H)^2 \cdot P(T)^4$$
$$= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{1}{64}.$$

So, the probability of getting exactly 2 heads in 6 tosses of a fair coin is ${}_6C_2 \cdot P(H)^2 \cdot P(T)^4 = 15 \cdot \frac{1}{64} = \frac{15}{64}$.

Solution 2 The 15 sequences with two heads are shown below.

HHTTTT	HTHTTT	HTTHTT	HTTTHT	HTTTTH
THHTTT	THTHTT	THTTHT	THTTTH	TTHHTT
TTHTHT	TTHTTH	TTTHHT	TTTHTH	TTTTHH

There are two outcomes for each of the flips (*H* or *T*), so the total number of sequences is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$. Thus, the probability of getting exactly two heads is $\frac{15}{64}$.

Connecting Probabilities with Combinations

Notice that the number of sequences of 2 heads in 6 flips equals the binomial coefficient $\binom{6}{2}$, which is the third number in row 6 of Pascal's Triangle. This is no accident: If an experiment is repeated n times , the number of possible sequences with r successes (and $n - r$ failures) is $\binom{n}{r}$, because determining such a sequence is equivalent to picking an r-element subset from $\{1, 2, \ldots, n\}$.

What happens if the two outcomes are not equally likely? Example 2 addresses this issue.

Example 2

Two generations ago, around 1950, the probability that a birth would be a multiple birth (twins, triplets, etc.) was about $\frac{1}{87}$. Mrs. Pereskier gave birth five times. Three of the births resulted in twins. What is the probability of this happening if multiple births occur at random?

The Multiple Birth Family Reunion in Mexico is an annual event for families of multiples.

Solution Let *M* be a multiple birth and *S* be a single birth. In this case, $P(M) = \frac{1}{87}$, so $P(S) = \frac{86}{87}$.

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There are $\binom{5}{3} = 10$ ways that 3 of the 5 births could be multiple births.

One of these ways yields the sequence SSMMM, which is what happened in the Pereskier family. The probability of this sequence is

$$\begin{aligned} P(S) \cdot P(S) \cdot P(M) \cdot P(M) \cdot P(M) &= (P(S))^2 \cdot (P(M))^3 \\ &= \left(\frac{86}{87}\right)^2 \cdot \left(\frac{1}{87}\right)^3 \\ &= \frac{7396}{4{,}984{,}209{,}207} \\ &\approx 0.00000148. \end{aligned}$$

There are 10 such sequences possible, so the probability of 3 multiple births in 5 births at that time was about $10 \cdot (0.00000148)$, or 0.0000148, or about 15 in a million.

This is a very low probability, and the actual occurrence of multiple births in some families is much higher than would be expected if they occurred randomly. This is how doctors realized that a tendency towards multiple births runs in some families.

STOP QY

Binomial Experiments

The situations of Examples 1 and 2 satisfy four criteria.

1. A task, called a **trial,** is repeated n times, where $n \geq 2$.
2. Each trial has outcomes that can be placed in one of only two categories, sometimes called "success" and "failure."
3. The trials are independent, that is, the probability of success on one trial is not affected by the results of earlier trials.
4. Each trial has the same probability of success.

When these four criteria are satisfied, the situation is called a **binomial experiment.** In a binomial experiment, the following properties hold:

- In n trials, there are $\binom{n}{r}$ possible sequences of r successes and $n - r$ failures.
- If the probability of success in any one trial is p, then the probability of failure is $q = 1 - p$ because success and failure are the only possible outcomes and they are mutually exclusive. Recall that categories are mutually exclusive if it is impossible for an element to belong to more than one of the categories.

QY

Refer to Example 2. What was the probability of exactly 1 multiple birth in 5 births?

- Because each trial has the same probability of success, the probability of any particular sequence of r successes and $n - r$ failures is $p^r q^{n-r}$.
- Because the trials are independent, the order in which successes and failures occur does not affect their probabilities. Therefore, the probability of *each* sequence of r successes and $n - r$ failures is $p^r q^{n-r}$, and their combined probability is $\binom{n}{r} p^r q^{n-r}$.

This argument proves the following theorem.

Binomial Probability Theorem

Suppose an experiment has an outcome with probability p, so that the probability the outcome does not occur is $q = 1 - p$. Then in n independent repetitions of the experiment, the probability that the outcome occurs r times is $\binom{n}{r} p^r q^{n-r}$.

These probabilities are often called *binomial probabilities* because of their connection with binomial coefficients.

GUIDED

Example 3

Suppose you roll two dice on each of three successive turns in a game. Compute the probability of rolling two sixes 0 times, 1 time, 2 times, and 3 times in those turns.

Solution Organize the computations in a table. Let rolling two sixes be a success and rolling anything else be a failure. If the die is fair, $P(\text{success})$ = probability of rolling a six on one die • probability of rolling a six on the other die $= \frac{1}{6} \cdot$ __?__ $=$ __?__. To compute the final result, substitute __?__ for p and $1 -$ __?__ $=$ __?__ for q into the Binomial Probability Theorem.

Number of Successes	Number of Failures	Binomial Expression	Probability
0	3	$\binom{3}{0} p^0 q^3$	__?__ ≈ __?__
1	2	$\binom{3}{1} p^1 q^{\underline{?}}$	__?__
2	1	__?__ $p^{\underline{?}} q^{\underline{?}}$	__?__
3	__?__	__?__	__?__

In Example 3, notice that the expressions in the Binomial Expression column are also the terms in the expansion of $(p + q)^3$. In general, the probability of r successes in n trials is the $p^r q^{n-r}$ term in the expansion of $(p + q)^n$. Because $p + q = 1$, $(p + q)^n = 1^n = 1$. That is, the sum of the probabilities of all possible outcomes computed using the Binomial Probability Theorem is 1.

Questions

COVERING THE IDEAS

In 1 and 2, a fair coin is flipped 5 times. A sequence of 5 H's (heads) and T's (tails) is recorded.

1. a. How many different sequences are possible with exactly 2 H's?
   b. How many different sequences are possible with exactly 2 T's?
   c. What is the probability of flipping exactly 2 heads?
2. a. How many different sequences are possible with exactly 4 T's?
   b. What is the probability of flipping exactly 4 tails?
3. About 1 in 35 births today is a multiple birth. Suppose there are 4 births in a family. What is the probability that exactly 1 of them is a multiple birth?

In 4 and 5, a fair coin is flipped 8 times. Give the probability of each event.

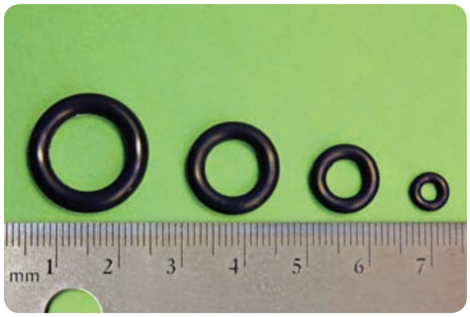
4. getting exactly 8 heads
5. getting exactly 4 heads
6. What is the probability of getting exactly r heads in n tosses of a fair coin?
7. Suppose that a coin is biased so that there is a 55% chance that the coin will show tails when tossed. Find the probability of each event.
   a. The coin shows heads when tossed.
   b. When tossed twice, the coin shows heads the first time and tails the second time.
   c. When tossed twice, the coin shows heads once and tails once.
8. Use the information in Question 3. What is the probability that, of the 3 births in a family today, at least 1 is a multiple birth?

In 9 and 10, suppose a stoplight is red for 45 seconds and green for 30 seconds. Suppose, also, that every day for a full week you get to this stoplight at a random time. What is the probability that the stoplight will be red

9. every day of the full week?
10. 3 of the 5 days of the work week?

The stoplight was invented by Garrett Augustus Morgan, Sr., an African-American born in Paris, Kentucky. He received a patent for the stoplight on November 20, 1923.

APPLYING THE MATHEMATICS



11. An O-ring is a circular mechanical seal (usually made of rubber) that generally prevents leakage between two compressed objects. A manufacturer of 14-mm diameter O-rings claims that 97.5% of the O-rings he manufactures are less than 14.5 mm in diameter. In a sample of 10 such O-rings, you find that three O-rings have a diameter greater than 14.5 mm. If the manufacturer's claim is correct, what is the probability of this outcome?

12. Slugger Patty McBattie has a batting average of 0.312. Use this average as her probability of getting a hit in a particular time at bat. In a game where she bats 5 times, what is the probability she gets exactly 2 hits?

In 13 and 14, suppose you have two minutes left to fill in the last 8 questions on a multiple-choice test. You can eliminate enough answers so that your probability of guessing the correct answer to any question is $\frac{1}{4}$.

13. What is the probability you get exactly 5 questions correct?

14. What is the probability you get 2 or more questions correct?

In 15 and 16, a student is given the quiz at the right.

15. Using R for right and W for wrong, list all possible ways the quiz might be answered. For example, getting all four right is coded RRRR. Assume an unanswered question is wrong.

16. Assuming that the student guesses on each item, and that the probability of guessing the right answer is $\frac{1}{2}$, calculate each probability.
    a. The student gets all 4 correct.
    b. The student gets exactly 2 correct.
    c. The student gets at least 2 correct.

QUIZ
1. Which is farther north, Anchorage, Alaska or Helsinki, Finland?
2. Is the 1,000,000th decimal place of π 5 or greater?
3. Is the area of Central Park in New York greater than 5 times the area of Hyde Park in London?
4. Did Euler die before or after Gauss was born?

REVIEW

17. Expand $(a + b)^5$. **(Lesson 13-6)**

18. Use the Binomial Theorem to approximate $(1.001)^{10}$ to fifteen decimal places. **(Lesson 13-6)**

19. The mean of three numbers in a geometric sequence is 35. The first number is 15. What might the other numbers be? **(Lessons 13-3, 7-5)**

20. A snail is crawling straight up a wall. The first hour it climbs 16 inches, the second hour it climbs 12 inches, and each succeeding hour it climbs $\frac{3}{4}$ the distance it climbed the previous hour. Assume this pattern holds indefinitely.

a. How far does the snail climb during the 7th hour?

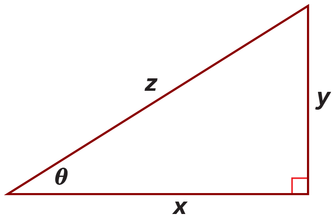
b. What is the total distance the snail has climbed in 7 hours? (**Lesson 13-2**)

In 21 and 22, expand and simplify. (**Lesson 11-2**)

21. $(t^2 - r^2)(t^2 + r^2)$

22. $(4a^2 + 2a + 1)(2a - 1)$

23. In the triangle at the right, find $\frac{\sin \theta}{\cos \theta}$. (**Lesson 10-1**)



24. Suppose an experiment begins with 120 bacteria and that the population of bacteria doubles every hour.

a. About how many bacteria will there be after 3 hours?

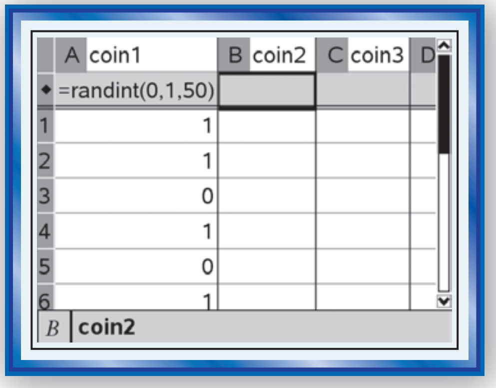
b. Write a formula for the number y of bacteria after x hours. (**Lesson 9-1**)

EXPLORATION

25. a. Use a random-number generator in a spreadsheet to simulate 50 trials of tossing 4 coins simultaneously. Record your results.

b. Calculate the relative frequency of each of the following outcomes: 0 heads, exactly 1 head, exactly 2 heads, exactly 3 heads, 4 heads.

c. How closely do your results agree with predictions based on the Binomial Probability Theorem? Do your think your random number generator program is a good one? Explain why or why not.



QY ANSWER

$\binom{5}{1} \cdot \left(\frac{1}{87}\right)^1 \cdot \left(\frac{86}{87}\right)^4 \approx 0.055$, or about 1 in 18