

Lesson

13-6

The Binomial Theorem

Vocabulary

binomial coefficients

► **BIG IDEA** The n th row of Pascal's Triangle contains the coefficients of the terms of $(a + b)^n$.

You have seen patterns involving squares of binomials in many places in this book. In this lesson we examine patterns involving the coefficients of higher powers.

Activity

MATERIALS CAS

Step 1 Expand the binomials in column 1 on a CAS and record the results in column 2 of a table like the one below.

Power of $(a + b)$	Expansion of $(a + b)^n$	Sum of Exponents of the Variables in Each Term
$(a + b)^0$	1	0
$(a + b)^1$	$a + b$	1
$(a + b)^2$	$a^2 + 2ab + b^2$	2
$(a + b)^3$?	?
$(a + b)^4$?	?
$(a + b)^5$?	?
$(a + b)^6$?	?

$$\text{expand}((a+b)^2) \quad a^2+2\cdot a\cdot b+b^2$$

Step 2 In column 3 of the table above, record the sum of the exponents of the variables in each term of the expansion of $(a + b)^n$.

Step 3 Set up a table like the one at the right. Record the coefficients of the terms in each expansion in column 2.

(continued on next page)

Power of $(a + b)$	Coefficients in the Expansion of $(a + b)^n$	Exponents of a in the Expansion	Exponents of b in the Expansion
$(a + b)^0$	1	0	0
$(a + b)^1$	1, 1	1, 0	0, 1
$(a + b)^2$	1, 2, 1	2, 1, 0	0, 1, 2
$(a + b)^3$?	?	?
$(a + b)^4$?	?	?
$(a + b)^5$?	?	?
$(a + b)^6$?	?	?

Mental Math

Expand.

- $(r - s)^2$
- $(7p + 9)^2$
- $(2ab - 3c)^2$
- $k^4(5m^2 + n^3)^2$

Step 4 What do you notice about the coefficients of the expansions of $(a + b)^n$?

Step 5 Write the coefficients of the expansion of $(a + b)^5$ using $\binom{n}{r}$ notation.

Step 6 Record the exponents of the powers of a and b in each term in the binomial expansions in the rightmost two columns of the table in Step 3.

Step 7 What do you notice about the exponents of a in each expansion of $(a + b)^n$? What do you notice about the exponents of b in each expansion of $(a + b)^n$?

The Activity reveals several properties of the expansion of $(a + b)^n$. Knowledge of these properties makes expanding $(a + b)^n$ easy.

- In each term of the expansion, the sum of the exponents of a and b is n .
- All powers of a occur in decreasing order from n to 0, while all powers of b occur in increasing order from 0 to n .
- If the power of b is r , then the coefficient of the term is $\binom{n}{r} = {}_n C_r$.

As a consequence of these properties, binomial expansions can be written using the $\binom{n}{r}$ symbol.

$$(a + b)^0 = \binom{0}{0}$$

$$(a + b)^1 = \binom{1}{0}a + \binom{1}{1}b$$

$$(a + b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$$

$$(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$\vdots \qquad \qquad \qquad \vdots$$

The information above is summarized in a famous theorem that was known to Omar Khayyam, the Persian poet, mathematician, and astronomer, who died around the year 1123. Of course, he did not have the notation we use today. Our notation makes it clear that the n th row of Pascal's triangle contains the coefficients of $(a + b)^n$.



Omar Khayyam

Binomial Theorem

For all complex numbers a and b , and for all integers n and r with $0 \leq r \leq n$,

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r.$$

A proof of the Binomial Theorem requires mathematical induction, a powerful proof technique beyond the scope of this book. You will see this proof in a later course.

Example 1

Expand $(a + b)^7$.

Solution First, write the powers of a and b in the form of the answer. Leave spaces for the coefficients.

$$(a + b)^7 = __ a^7 + __ a^6 b + __ a^5 b^2 + __ a^4 b^3 + __ a^3 b^4 + __ a^2 b^5 + __ a b^6 + __ b^7$$

Second, fill in the coefficients using $\binom{n}{r}$ notation.

$$(a + b)^7 = \binom{7}{0} a^7 + \binom{7}{1} a^6 b + \binom{7}{2} a^5 b^2 + \binom{7}{3} a^4 b^3 + \binom{7}{4} a^3 b^4 + \binom{7}{5} a^2 b^5 + \binom{7}{6} a b^6 + \binom{7}{7} b^7$$

Finally, evaluate the coefficients, either by referring to row 7 of Pascal's Triangle or by using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$$(a + b)^7 = a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$$

The Binomial Theorem can be used to expand a variety of expressions.

Example 2

Expand $(3x - 4y)^3$.

Solution 1 The expansion follows the form of $(a + b)^3$.

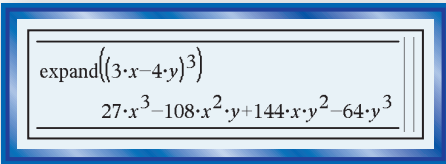
$$(a + b)^3 = \binom{3}{0} a^3 + \binom{3}{1} a^2 b + \binom{3}{2} a b^2 + \binom{3}{3} b^3.$$

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Think of $3x$ as a and $-4y$ as b and substitute.

$$\begin{aligned}(3x - 4y)^3 &= 1(3x)^3 + 3(3x)^2(-4y) + 3(3x)(-4y)^2 + 1(-4y)^3 \\ &= 27x^3 - 108x^2y + 144xy^2 - 64y^3\end{aligned}$$

Solution 2 Expand the binomial on a CAS.



A screenshot of a computer algebra system (CAS) interface. It shows the command 'expand((3*x-4*y)^3)' and the resulting output: '27*x^3-108*x^2*y+144*x*y^2-64*y^3'.

GUIDED

Example 3

Expand $(2x^2 + 1)^4$.

Solution Think of $2x^2$ as a and 1 as b . Then follow the form of $(a + b)^4$.

$$\begin{aligned}(2x^2 + 1)^4 &= (\underline{\quad})(2x^2)^{\underline{\quad}} + (\underline{\quad})(2x^2)^{\underline{\quad}} \cdot (1)^{\underline{\quad}} + \\ &\quad (\underline{\quad})(2x^2)^{\underline{\quad}} \cdot (1)^{\underline{\quad}} + (\underline{\quad})(2x^2)^{\underline{\quad}} \cdot (1)^{\underline{\quad}} + \\ &\quad (\underline{\quad}) \cdot (1)^{\underline{\quad}} \\ &= \underline{\quad}x^{\underline{\quad}} + \underline{\quad}x^{\underline{\quad}} + \underline{\quad}x^{\underline{\quad}} + \underline{\quad}x^{\underline{\quad}} + \underline{\quad}\end{aligned}$$

Check Substitute a value for x in the binomial power and in the expansion. The two results should be equal.

You can also use the Binomial Theorem to quickly find any term in the expansion of a binomial power without writing the full expansion.

Example 4

Find the 8th term in the expansion of $(a + b)^{20}$.

Solution The formula $(a + b)^{20} = \sum_{r=0}^{20} \binom{20}{r} a^{20-r} b^r$ gives the full

expansion of the binomial. Because r starts at 0, the 8th term is when $r = 7$.

$$\binom{20}{7} a^{20-7} b^7 = 77,520 a^{13} b^7$$

The 8th term in the expansion is $77,520a^{13}b^7$.

STOP QY

Due to their use in the Binomial Theorem, the numbers in Pascal's Triangle are sometimes called **binomial coefficients**. The Binomial Theorem has a surprising number of applications in estimation, counting problems, probability, and statistics. You will study these applications in the remainder of this chapter.

QY

Find the 13th term in the expansion of $(x + 2)^{15}$.

Questions

COVERING THE IDEAS

- Expand $(x + y)^2$.
 - What are the coefficients of the terms in the expansion of $(x + y)^3$?

In 2–4, expand each binomial power.

$$2. (a - 3b)^3 \qquad 3. \left(\frac{1}{2} - m^2\right)^4 \qquad 4. (2x + 5y)^3$$

In 5 and 6, find the 5th term in the binomial expansion.

$$5. (x + y)^{10} \qquad 6. (a - 3b)^8$$

In 7 and 8, find the second-to-last term in the binomial expansion. (This term is called the *penultimate* term.)

$$7. (5 - 2n)^9 \qquad 8. (3j + k)^{12}$$

APPLYING THE MATHEMATICS

In 9 and 10, convert to an expression in the form $(a + b)^n$.

$$9. \sum_{r=0}^{14} \binom{14}{r} x^{14-r} 2^r \qquad 10. \sum_{i=0}^n \binom{n}{i} y^{n-i} (-3w)^i$$

- Multiply the binomial expansion for $(a + b)^3$ by $a + b$ to check the expansion for $(a + b)^4$.
- Multiply and simplify $(a^2 + 2ab + b^2)(a^2 + 2ab + b^2)$.
 - Your answer to Part a should be a power of $a + b$. Which one? Explain your answer.

In 13 and 14, use this information. The Binomial Theorem can be used to approximate some powers quickly without a calculator. Here is an example.

$$\begin{aligned} (1.002)^3 &= (1 + 0.002)^3 \\ &= 1^3 + 3 \cdot 1^2 \cdot (0.002) + 3 \cdot 1 \cdot (0.002)^2 + (0.002)^3 \\ &= 1 + 0.006 + 0.000012 + 0.000000008 \\ &= 1.006012008 \end{aligned}$$

Because the last two terms in the expansion are so small, you may ignore them in an approximation. So $(1.002)^3 \approx 1.006$ to the nearest thousandth.

- Show how to approximate $(1.003)^3$ to the nearest thousandth without a calculator. Check your answer with a calculator.
- Show how to approximate $(1.001)^4$ to nine decimal places without a calculator.

15. a. Evaluate 11^0 , 11^1 , 11^2 , 11^3 , and 11^4 . How are these numbers related to Pascal's Triangle?
 b. Expand $(10 + 1)^4$ using the Binomial Theorem.
 c. Use the Binomial Theorem to calculate 11^5 .

REVIEW

True or False In 16 and 17, explain your reasoning.

16. $\binom{99}{17}$ is an integer. (Lesson 13-5)
17. $\frac{n!}{(n-2)!}$ is always an integer when $n \geq 2$. (Lesson 13-4)
18. Simplify: ${}_9C_0 + {}_9C_1 + {}_9C_2 + {}_9C_3 + {}_9C_4 + {}_9C_5 + {}_9C_6 + {}_9C_7 + {}_9C_8 + {}_9C_9$. (Lesson 13-4)
19. Consider the ellipse with equation $\frac{x^2}{15} + \frac{y^2}{26} = 1$. (Lessons 12-5, 12-4)
- Give the length of its major axis.
 - Give the coordinates of the endpoints of its major and minor axes.
 - Find the coordinates of its foci F_1 and F_2 .
 - If P is a point on this ellipse, find $PF_1 + PF_2$.
 - Find the area of the ellipse.
20. Paola has been saving to buy a condo for five years. At the beginning of the first year, she placed \$2200 in a savings account that pays 3.7% interest annually. At the beginning of the second, third, fourth, and fifth years, she deposited \$2350, \$2125, \$2600, and \$2780, respectively, into the same account. At the end of the five years, does Paola have enough money in the account to make a \$15,000 down payment? If not, how much more does she need? (Lesson 11-1)

In 21–24, solve. (Lessons 9-9, 9-7, 9-5)

21. $\log_7 y = 2 \log_7 13$ 22. $2 \ln 13 = \ln x$
 23. $\log z = 5$ 24. $\ln(3x) = \ln 2 + \ln 18$

EXPLORATION

25. The expansion of $(a + b)^3$ has 4 terms.
- How many terms are in the expansion of $(a + b + c)^3$?
 - How many terms are in the expansion of $(a + b + c + d)^3$?
 - Generalize these results.

QY ANSWER

$$\binom{15}{12} x^{15-12} \cdot 2^{12} =$$

$$1,863,680x^3$$