Lesson 13-5 Pascal's Triangle

Vocabulary

Pascal's Triangle

BIG IDEA The *n*th row of Pascal's Triangle contains the number of ways of choosing *r* objects out of *n* objects without regard to their order, that is, the number of combinations of *r* objects out of *n*.

Mental Math

Let *D* be a relation that maps any polynomial onto its degree.

a. Name three ordered pairs in *D*.

b. Is *D* a function?

c. What is the range of *D*?

d. Is the inverse of *D* a function?

Very often a single idea has applications to many parts of mathematics. On the first page of this chapter, we mentioned *Pascal's Triangle*. Pascal's Triangle is not a triangle in the geometric sense. It is an infinite array of numbers in a triangular shape. The top rows of the triangle are shown below.

1	row 0
1 1	row 1
1 2 1	row 2
1 3 3 1	row 3
1 4 6 4 1	row 4
1 5 10 10 5 1	row 5
1 6 15 20 15 6 1	row 6
1 7 21 35 35 21 7 1	row 7

Pascal's Triangle

This array seems to have first appeared in the 11th century in the works of Abu Bakr al-Karaji, a Persian mathematician, and Jia Xian, a Chinese mathematician. The works of both of these men are now lost, but 12th-century writers refer to them. Versions of the array were discovered independently by the Europeans Peter Apianus in 1527 and Michael Stifel in 1544. But in the western world the array is known as Pascal's Triangle after Blaise Pascal (1623–1662), the French mathematician and philosopher who discovered many properties relating the numbers in the array. Pascal himself called it the *triangle arithmetique*, which literally translates as the "arithmetical triangle."

How Is Pascal's Triangle Formed?

Pascal's Triangle is formed in a very simple way. You can think of Pascal's Triangle as a two-dimensional sequence in which each element is determined by a row and its position in that row.

Here is a recursive definition for the sequence: The only element in the top row (row 0) is 1. The first and last elements of all other rows are also 1. If x and y are located next to each other in a row, the element just below and directly between them is $x + y$, as illustrated at the right.

x y
$x + y$

For instance, from row 7 you can get row 8 as follows.

1 7 21 35 35 21 7 1   row 7
1 8 28 56 70 56 28 8 1   row 8

From this recursive definition, you can obtain any row in the array if you know the preceding row.

Example 1

Write row 9 of Pascal's Triangle.

Solution Begin by listing row 8, as shown above. Apply the recursive definition to generate row 9. Remember that the first and last elements of each row are 1.

1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

STOP QY1

> **QY1**
> Write row 10 of Pascal's Triangle.

The elements in the nth row of Pascal's triangle are identified as $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$, ..., $\binom{n}{n}$. The top row of the array is called row 0, so in this row $n = 0$. It has one element, its first element, $\binom{0}{0}$. So, $\binom{0}{0} = 1$. The two elements of row 1 are $\binom{1}{0}$ and $\binom{1}{1}$. The three elements of row 2 are $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$. In general, the $(r + 1)$st element in row n of Pascal's triangle is denoted by $\binom{n}{r}$.

STOP QY2

> **QY2**
> Write row 8 of Pascal's Triangle using $\binom{n}{r}$ notation.

On the previous page, we wrote a recursive definition of Pascal's Triangle in words. Now, we can write a recursive definition using the $\binom{n}{r}$ symbol. The recursive rule involves two variables because Pascal's Triangle is a two-dimensional sequence, that is, a sequence in two directions: down and across.

Definition of Pascal's Triangle

Pascal's Triangle is the sequence satisfying

1. $\binom{n}{0} = \binom{n}{n} = 1$, for all integers $n \geq 0$ and
2. $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$, for $0 \leq r < n$.

Part 1 of the definition gives the "sides" of the triangle. Part 2 is a symbolic way of stating that adding two adjacent elements in one row gives an element in the next row.

Example 2

Find a solution to the equation $\binom{x}{y} = \binom{7}{5} + \binom{7}{6}$.

Solution 1 Apply Part 2 of the definition of Pascal's Triangle.

Here $n = 7$ and $r = 5$. Substituting these values into Part 2, we have

$$\binom{7}{5} + \binom{7}{6} = \binom{8}{6}.$$

So, $x = 8$ and $y = 6$.

Solution 2 Find $\binom{7}{5}$ and $\binom{7}{6}$ in Pascal's triangle and add the results.

$\binom{7}{5}$ is the 6th element in row 7. So $\binom{7}{5} = 21$. $\binom{7}{6}$ is the 7th element in row 7, so $\binom{7}{6} = 7$. $21 + 7 = 28$.

Locate where 28 appears in Pascal's Triangle.

28 is the 3rd element in row 8 and the 7th element in row 8. So, $x = 8$ and $y = 2$ is one solution, and $x = 8$ and $y = 6$ is another solution.

Entries in Pascal's Triangle

There is a very close connection between combinations and the elements in the rows of Pascal's Triangle.

Activity

Step 1 Calculate ${}_4C_0$, ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, and ${}_4C_4$.

Step 2 How are the results of Step 1 related to Pascal's Triangle?

Step 3 Find ${}_7C_0$, ${}_7C_1$, ${}_7C_2$, ${}_7C_3$, ${}_7C_4$, ${}_7C_5$, ${}_7C_6$, and ${}_7C_7$.

Step 4 How are the results of Step 3 related to Pascal's Triangle?

Step 5 Generalize Steps 1–4.

The generalization of the Activity is stated below. It was first proved by the famous English mathematician and physicist Isaac Newton in the 17th century.

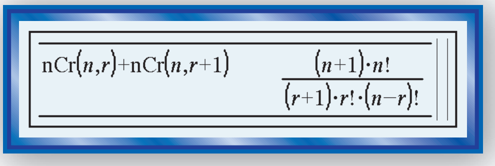
Pascal's Triangle Explicit Formula

If n and r are integers with $0 \le r \le n$, then $\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$.

Proof To show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, it is enough to show that the factorial expression $\frac{n!}{r!(n-r)!}$ satisfies the relationships involving $\binom{n}{r}$ in the recursive definition of Pascal's Triangle.

(1) When $n \ge 0$, does the expression $\frac{n!}{0!(n-0)!}$ equal the expression $\frac{n!}{n!(n-n)!}$ and equal 1? Yes, since $\frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{n!}{1 \cdot n!} = 1$, and $\frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n! \cdot 1} = 1$.
Thus, the formula works for the "sides" of Pascal's triangle.

(2) To prove that the expression $\frac{(n+1)!}{(r+1)!(n-r)!}$ is the sum of the expressions $\frac{n!}{r!(n-r)!}$ and $\frac{n!}{(r+1)!(n-r-1)!}$, use a calculator or CAS. Enter ${}_nC_r + {}_nC_{r+1}$.



This CAS displays $\frac{(n+1) \cdot n!}{(r+1) \cdot r! \cdot (n-r)!}$.

Using the definition of factorial,

$$\frac{(n+1) \cdot n!}{(r+1) \cdot r! \cdot (n-r)!} = \frac{(n+1)!}{(r+1)!(n-r)!}.$$

The right side is an expression for $\binom{n+1}{r+1}$. So, this explicit formula gives the same sequence as the recursive formula that defines Pascal's Triangle.

The result of all this is an exceedingly useful fact: *The elements in row n of Pascal's triangle are the numbers of combinations possible from n things taken 0, 1, 2, …, n at a time.* So, you do not need to calculate all the rows of Pascal's triangle to get the next row. You can use your knowledge of combinations.

GUIDED

Example 3

Find $\binom{8}{5}$.

Solution 1 Use the Pascal's Triangle Explicit Formula.

$$\binom{8}{5} = \frac{?!}{?!(?-?)!} = \underline{\ ?\ }$$

Solution 2 $\binom{8}{5}$ is the _?_th element in row _?_ of Pascal's triangle. From the second page of the lesson, it is _?_.

Questions

COVERING THE IDEAS

1. When and where did the array known as Pascal's Triangle first appear?
2. When and where did Pascal live?
3. Explain how entries in a row of Pascal's Triangle can be used to obtain entries in the next row.
4. Write row 11 of Pascal's Triangle.
5. Write row 5 of Pascal's Triangle using $\binom{n}{r}$ notation.
6. **Fill in the Blanks** The element $\binom{16}{8}$ is the _?_ element in row _?_ of Pascal's Triangle.

In 7–12, calculate the element of Pascal's Triangle and give its location in the triangle.

7. $\binom{6}{4}$
8. $\binom{12}{3}$
9. ${}_{10}C_0$
10. $\binom{8}{8}$
11. $\binom{7}{5}$
12. ${}_{18}C_{10}$
13. Calculate the 3rd element in the 100th row of Pascal's triangle.

APPLYING THE MATHEMATICS

14. Simplify $\binom{n}{n-1}$.

In 15 and 16, find a solution to the equation.

15. $\binom{9}{4} + \binom{9}{5} = \binom{x}{y}$

16. $\binom{11}{5} + \binom{a}{b} = \binom{12}{5}$

17. a. Find the sum of the elements in each of the rows 1 through 6 of Pascal's triangle.

    b. Based on the results of Part a, what do you think is the sum of the elements in row 7?

    c. Write an expression for the sum of the elements in row n of Pascal's Triangle.

18. What sequence is defined by the second element of each row in Pascal's Triangle?

19. Where in Pascal's Triangle can the sequence of triangular numbers 1, 3, 6, 10, 15, ... $\frac{n(n+1)}{2}$, … be found?

20. Where in Pascal's Triangle can the sequence of series with terms the first n triangular numbers 1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10, ... be found?

REVIEW

21. How many golf foursomes can be formed from twelve people? **(Lesson 13-4)**

22. Consider drawing cards at random from a standard 52-card deck.

    a. In how many ways can you draw one card?

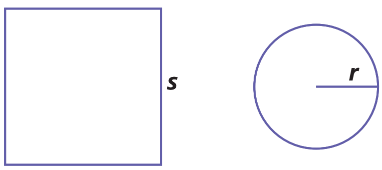
    b. How many different pairs of cards can you draw if order does not matter?

    c. What are the chances of drawing the four aces on your first 4 draws? **(Lesson 13-4)**

23. Will had a mean of 86 on five tests. After the lowest test score was dropped, his mean was 93. What was the score that was dropped? **(Lesson 13-3)**

24. Give the first 8 terms of the sequence $a_n = \sin(n \cdot 45°)$. **(Lessons 10-1, 1-8)**

25. A square has double the area of a circle. The square has side length 10. What is the radius of the circle? **(Lesson 6-2)**



26. Expand. **(Lesson 6-1)**

a. $(a + b)^2$

b. $3(x - 7)^2$

c. $(3p + 6q)^2 - (3p - 6q)^2$

27. Explain why $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ does not have an inverse. **(Lesson 5-5)**

In 28–30 solve. (Lesson 1-6)

28. $3y + 60 = 5y + 42$

29. $\frac{4}{y} + \frac{8}{y} = 5$

30. $0.05x + 0.1(2x) + 0.25(100 - 3x) = 20$

EXPLORATION

31. There are six elements surrounding each element not on a side of Pascal's Triangle. For instance, around 15 in row 6 are the elements 5, 10, 20, 35, 21, and 6. In 1969, an amazing property about the product of these six elements was discovered by Verner Hoggatt and Walter Hansell of San Jose State University.

1	5	10	10	5	1			row 5
1	6	15	20	15	6	1		row 6
1	7	21	35	35	21	7	1	row 7

a. Find the product of all the elements surrounding the number 15 in row 6.

b. Repeat for the number 3 in row 3.

c. Find the surrounding product for each element not on a side of row 5 of the triangle.

d. Describe the pattern you see in your products in Parts a through c.

e. Support your answer to Part d by calculating the product for two elements in row 8 of the triangle.

QY ANSWERS

1. 1 10 45 120 210 252 210 120 45 10 1

2. $\binom{8}{0}$ $\binom{8}{1}$ $\binom{8}{2}$ $\binom{8}{3}$ $\binom{8}{4}$ $\binom{8}{5}$ $\binom{8}{6}$ $\binom{8}{7}$ $\binom{8}{8}$