

Lesson

13-4

Subsets and
Combinations

Vocabulary

permutation

!, factorial symbol

 $n!$

combination

► **BIG IDEA** Given a set of n objects, there are formulas for the number of ways of choosing r objects where the order or the objects matters, and for the number of ways of choosing r objects without regard to their order.

Permutations

An arrangement of objects where order matters is called a **permutation**. With 3 objects A , B , and C , there are 6 possible permutations: ABC , ACB , BAC , BCA , CAB , and CBA . You can think of these 6 permutations in many ways, such as ways to arrange 3 objects on a shelf or orders in which runners could win medals in an Olympic race.

Example 1

- Write all the possible orders in which 4 runners A , B , C , and D might finish a race.
- How many permutations of 4 runners are there?



Solution

- Make a list as shown below. Assume A finishes first. The left column lists the 6 possible orders of B , C , and D finishing behind A . The next column has B first followed by the 6 possible orders of A , C , and D . The third and fourth columns begin with C and D , respectively.

$ABCD$	$BACD$	$CABD$	$DABC$
$ABDC$	$BADC$	$CADB$	$DACB$
$ACBD$	$BCAD$	$CBAD$	$DBAC$
$ACDB$	$BCDA$	$CBDA$	$DBCA$
$ADBC$	$BDAC$	$CDAB$	$DCAB$
$ADCB$	$BDCA$	$CDBA$	$DCBA$

- Count the permutations you listed. There are 24 permutations. Notice that the number of permutations of 4 objects is 4 times the number of permutations of 3 objects, or $4 \cdot 6$.

Mental Math

Simplify.

a. $\{n \mid n < 3\} \cup \{n \mid n \geq -8\}$

b. $\{p \mid p \text{ is divisible by } 2\} \cap \{p \mid p \text{ is divisible by } 3\}$

c. the set of all circles \cup the set of all ellipses

d. the set of all rectangles with perimeter 50 \cap the set of all squares with area 225

Special Olympics serves people with intellectual disabilities in over 180 countries.

To list the possible ways in which 5 people could finish a race, you could begin with the list in Example 1. Call the fifth racer E . In each permutation in the list, you can insert E in 5 places: at the beginning, in one of the three middle spots, or at the end. For instance, inserting E into $ABCD$ yields $EABCD$, $AEBCD$, $ABECD$, $ABCED$, or $ABCDE$. This means that the number of permutations of 5 objects is 5 times the number of permutations of 4 objects, or $5 \cdot 24$.

The Factorial Symbol

You may have noticed a pattern. The number of permutations of 2 objects A and B is 2, AB and BA , and $2 = 2 \cdot 1$. The number of permutations of 3 objects A , B , and C is 6, which is $3 \cdot 2$, or $3 \cdot 2 \cdot 1$. The number of permutations of 4 objects is $4 \cdot 6$, or $4 \cdot 3 \cdot 2 \cdot 1$. The number of permutations of 5 objects is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 120.

These products of the integers n through 1 are represented by a special symbol, called the *factorial symbol*. The **factorial symbol**, $!$, is an exclamation point, and $n!$ is read “ n factorial.”

Definition of Factorial

Let n be any integer ≥ 2 . Then $n!$ is the product of the integers from 1 through n .

A generalization of Example 1 can be described using factorials.

Number of Permutations Theorem

There are $n!$ permutations of n distinct objects.

In the order of operations, factorials are calculated before multiplications or divisions. That is, $2 \cdot 5! = 2 \cdot 120 = 240 \neq 10!$.

Activity

Step 1 Copy and fill in the table at the right.

Step 2 Describe the pattern you see in the table. Use the pattern to write a recursive formula for the sequence $f_n = n!$.

n	3	4	5	6	7	8	9	10
$n!$?	?	120	?	?	?	362,880	?
$(n-1)!$	2	?	?	?	720	?	?	?
$\frac{n!}{(n-1)!}$?	?	?	?	?	?	?	?

The pattern in the Activity is a fundamental property of factorials.

Factorial Product Theorem

For all $n \geq 1$, $n! = n \cdot (n-1)!$.

When $n \geq 3$, the theorem follows from the definition of factorial. For the theorem to hold when $n = 2$, we must have $2! = 2 \cdot (2 - 1)! = 2 \cdot 1!$. This means that $1!$ has to equal 1. This makes sense with permutations. If there is only one object, there is only one order. If the theorem is to hold when $n = 1$, then it must be that $1! = 1 \cdot (1 - 1)! = 1 \cdot 0!$. This means that we must have $0! = 1$.

Many calculators and CAS give exact values of $n!$ for small values of n , but for larger values, they give approximations in scientific notation. For instance, when $20!$ is entered, one calculator displays

2432902008176640000

while another displays $2.4329 \text{ E } 18$, which means 2,432,900,000,000,000,000.

Products of Consecutive Integers

Factorials help you calculate products of consecutive integers, starting at any number.

Example 2

Find $7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16$ using factorials.

Solution 1 Multiply the given product by a factorial so that the final product is a factorial.

$$\text{Let } x = 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16.$$

$$\text{Notice that } 6! \cdot x = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot x = 16!.$$

$$\text{Solving for } x, \quad x = \frac{16!}{6!} = 29,059,430,400.$$

Solution 2 Multiply the given product by $\frac{6!}{6!}$. This does not change its value.

$$\begin{aligned} 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \\ &= \frac{6!}{6!} \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \\ &= \frac{16!}{6!} = 29,059,430,400 \end{aligned}$$

STOP QY1

▶ QY1

Write $22 \cdot 23 \cdot 24$ as a quotient of two factorials.

Subsets and Combinations

You can apply the technique in Example 2 to problems where you are choosing subsets and order does not matter.

Example 3

A committee of 4 people is to be chosen from 10 applicants. In how many different ways can this be done?

Solution Think of the applicants as the set $\{A, B, C, D, E, F, G, H, I, J\}$. Each possible committee is a 4-element subset of this set. For instance, two possible committees are $\{D, C, A, B\}$ and $\{C, D, F, J\}$.

Form the committees one person at a time. There are 10 possibilities for the first person. After selecting the first person, there are 9 possibilities for the second person. After selecting the first two people, there are 8 possibilities for the third person. After selecting the first three people, there are 7 possibilities for the fourth person. So it seems that there are $10 \cdot 9 \cdot 8 \cdot 7$ possible committees.

However, this assumes that the order in which the people are chosen makes a difference, but the order of people in a committee does not matter: $\{B, E, H, I\}$ and $\{H, B, E, I\}$ are the same committee. In fact, there are $4! = 24$ different orders of the elements B, E, H, and I, all of which form the same committee. So, the answer $10 \cdot 9 \cdot 8 \cdot 7$ is $4!$ times what you need.

The number of committees with 4 people is $\frac{10 \cdot 9 \cdot 8 \cdot 7}{4!}$.

Multiply both the numerator and denominator by $6!$.

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} \cdot \frac{6!}{6!} = \frac{10!}{4! \cdot 6!} = 210$$

So, there are 210 ways to choose a committee of 4 from a set of 10 people.

**STOP QY2**

Example 3 can be viewed as a problem in counting subsets. How many subsets of 4 elements are possible from a set of 10 elements? It also can be viewed as a problem in counting *combinations* of objects. How many combinations of 4 objects are possible from 10 different objects?

Any choice of r objects from n objects *when the order of choice does not matter* is called a **combination**. The number of combinations of r objects that can be created from n objects is denoted ${}_n C_r$. The following theorem connects combinations with counting subsets.

Combination Counting Formula

The number ${}_n C_r$ of subsets, or combinations, of r elements that can be formed from a set of n elements is given by the formula

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

QY2

How many committees of 3 people can be chosen from 10 applicants?

Proof There are n choices for the first element in a subset. Once that element has been picked, there are $n - 1$ choices for the second element, and $n - 2$ choices for the third element. This continues until all r elements have been picked. There are $(n - r + 1)$ choices for the r th element.

So, if all possible orders are considered different, there are

$$\underbrace{n(n-1)(n-2) \dots (n-r+1)}_{r \text{ factors}}$$

ways to choose them. But each subset is repeated $r!$ times with the same elements in various orders. So the number of different subsets is

$${}_n C_r = \frac{n(n-1)(n-2) \dots (n-r+1)}{r!}.$$

Multiplying both numerator and denominator by $(n - r)!$ gives the Combination Counting Formula.

${}_n C_r$ is sometimes read “ n choose r .” Another notation for the number of combinations is $C(n, r)$. Both have the same meaning and are equal to $\frac{n!}{r!(n-r)!}$. Example 3 shows that ${}_{10}C_4 = C(10, 4) = 210$. Many calculators have keys that enable you to calculate ${}_n C_r$ directly.

GUIDED

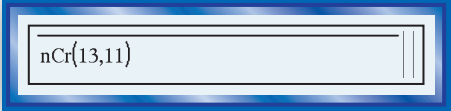
Example 4

How many subsets of 11 elements are possible from a set of 13 elements?

Solution Evaluate $C(13, 11)$. Use the Combination Counting Formula with $n = \underline{\quad ? \quad}$ and $r = \underline{\quad ? \quad}$.

$$\text{Then } \frac{n!}{r!(n-r)!} = \frac{?}{? \cdot ?} = \underline{\quad ? \quad}.$$

Check Use a calculator or CAS. At the right is one way to enter the combination.



$nCr(13,11)$

Example 5

Given 7 points in a plane, with no 3 of them collinear, how many different triangles can have 3 of these points as vertices?

Solution 1 Because no 3 points are collinear, any choice of 3 points from the 7 points determines a triangle. Use the Combination Counting Formula with $n = 7$ and $r = 3$.

$$\text{The number of possible triangles is } {}_7 C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = 35.$$

Solution 2 Use the idea of the proof of the Combination Counting Formula.

(continued on next page)

The first vertex of the triangle can be chosen in 7 ways. The second vertex can then be chosen in 6 ways. And the third vertex can then be chosen in 5 ways. So, if order mattered, there would be $7 \cdot 6 \cdot 5 = 210$ different triangles. But order doesn't matter and each triangle is counted $3! = 6$ times. So divide 210 by 6, giving 35 different triangles.

Questions

COVERING THE IDEAS

- Write all permutations of the three symbols P, R, M .
 - Write all combinations of the three symbols P, R, M .
- How many permutations are there for the 5 vowels A, E, I, O, and U?
- Give the values of $1!, 2!, 3!, 4!, 5!, 6!$, and $7!$.
- Explain why $23! = 23 \cdot 22!$.
- Explain, in words, the difference between a permutation and a combination.
- Write $100 \cdot 101 \cdot 102 \cdot 103$ as the quotient of two factorials.
- How many combinations of r objects can you make from n different objects?
- What is another way to represent ${}_n C_r$?
- A cone of 3 different scoops of ice cream is to be chosen from 5 different flavors. In how many ways can this be done?
- How many subcommittees of 6 people are possible in a committee of 15?
- Refer to Example 5. How many different line segments can have 2 of the 7 points as endpoints?

APPLYING THE MATHEMATICS

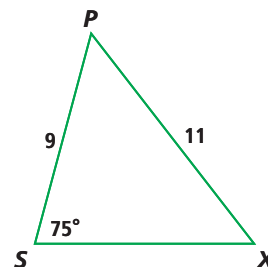
- Prove that $(n + 1)! = n!(n + 1)$.
- Recall that the U.S. Congress consists of 100 senators and 435 representatives.
 - How many four-person senatorial committees are possible?
 - How many four-person house committees are possible?
 - A “conference committee” of 4 senators and 4 representatives is chosen to work out differences in bills passed by the two houses. How many different conference committees are possible?



14. Consider a set of n elements.
- How many subsets of any number of elements are possible when $n = 1, 2, 3, 4, 5$?
 - Based on your answers to Part a, make a conjecture about the number of subsets for a set with k elements.
15. Dyana sells custom-made tie-dyed T-shirts. A customer chooses 6 dyes from 25 possibilities. Dyana advertises that she offers 150,000 different dye combinations.
- Assuming a customer chooses 6 dyes, how many dye combinations are possible? Is the advertisement correct?
 - How many choices does a customer have if the order of color choice matters?

REVIEW

16. Find the standard deviation of the data set $\{4, 11, 25, 39, 39, 25, 11, 4\}$. (Lesson 13-3)
17. If two sets of scores have the same mean but the standard deviation of the first set is much larger than that of the second set, what can you conclude? (Lesson 13-3)
18. In $\triangle SPX$, $m\angle S = 75^\circ$, $s = 11$, and $x = 9$. Find $m\angle X$. (Lesson 10-7)
19. a. Find an equation for the inverse of the linear function with equation $y = mx + b$.
- b. How are the slopes of a linear function and its inverse related?
- c. When is the inverse not a function? (Lessons 8-2, 1-4)
20. Suppose y varies inversely as the cube of w . If y is 6 when w is 5, find y when w is 11. (Lesson 2-2)
21. Give a set of integers whose mean is 12, whose mode is 14, and whose median is 13. (Previous Course)



EXPLORATION

22. a. Consider the state name MISSISSIPPI. In how many different ways can you rearrange the letters, if you do not distinguish between letters that are the same?
- b. Repeat Part a using WOOLLOOMOOLOO, the name of a town near Sydney, Australia.

QY ANSWERS

- $22 \cdot 23 \cdot 24 = \frac{24!}{21!}$
- $\frac{10 \cdot 9 \cdot 8}{3!} \cdot \frac{7!}{7!} = \frac{10!}{3! \cdot 7!} = 120$