

Lesson

13-2

Geometric Series

Vocabulary

geometric series

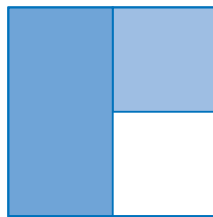
BIG IDEA There are several ways to find the sum of the successive terms of a finite geometric sequence.

Activity

Step 1 Draw a large square on a sheet of paper.

Step 2 Divide the square into two equal parts and shade one of the regions. How much of the square has been shaded?

Step 3 Divide the unshaded half into two equal parts and shade one of the regions. How much of the original square does this region represent? How much of the total square have you shaded? The figure at the right shows one possible result of Steps 1–3.



Step 4 Repeat Step 3 three more times. Fill in a table like the one below where n represents the number of shaded regions and S_n is the total fraction of the original large square that is shaded.

n	1	2	3	4	5
g_n (series)	$\frac{1}{2}$	$\frac{1}{2} + \frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?	?
S_n (value)	$\frac{1}{2}$	$\frac{3}{4}$?	?	?

Step 5 What is the value of S_6 ? Describe the n th term of the series and the value of S_n .

Step 6 Will the original square ever be entirely shaded? Explain why or why not.

In the Activity, the terms in the series g form a geometric sequence with first term $\frac{1}{2}$ and constant ratio $\frac{1}{2}$. So the k th term in the sequence is $\left(\frac{1}{2}\right)^k$. For instance, the 5th term in the sequence is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$, and

$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \sum_{k=1}^5 \frac{1}{2^k} = \frac{31}{32}.$$

Mental Math

Is the point (2, 1) included in the set?

- the circle with equation $x^2 + y^2 = 5$
- the interior of the circle with equation $x^2 + y^2 = 5$
- the exterior of the circle with equation $x^2 + y^2 = 3$
- the interior of the circle with equation $(x - 1)^2 + (y + 1)^2 = 3$

GUIDED

Example 1

- Write the value of S_{10} from the Activity as a sum of terms of a geometric sequence.
- Write S_{10} using Σ -notation.
- Compute exact and approximate values of S_{10} using a calculator or CAS.

Solution

- The terms in the geometric sequence are the first ten positive integer powers of $\frac{1}{2}$. So, $S_{10} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad}$.
- There are 10 terms and an expression for the k th term is $\left(\frac{1}{2}\right)^k$.
So $S_{10} = \sum_{\underline{\quad}}^{\underline{\quad}} \underline{\quad}$.
- The exact value is $\underline{\quad}$. An approximate value is $\underline{\quad}$.

An indicated sum of successive terms of a geometric sequence, like the one for S_{10} in Part a of Example 1, is called a **geometric series**. As with arithmetic series, there are formulas for the values of geometric series.

A Formula for the Value of Any Finite Geometric Series

In Example 1, notice that if each term of the sequence is halved, many values are identical to those in the original series:

$$S_{10} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \left(\frac{1}{2}\right)^9 + \left(\frac{1}{2}\right)^{10}$$

$$\frac{1}{2}S_{10} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \left(\frac{1}{2}\right)^9 + \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{11}$$

Subtracting the second equation from the first yields

$$\frac{1}{2}S_{10} = \frac{1}{2} + \left(\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{1}{8} - \frac{1}{8}\right) + \dots + \left(\left(\frac{1}{2}\right)^{10} - \left(\frac{1}{2}\right)^{10}\right) - \left(\frac{1}{2}\right)^{11}$$

That is, $\frac{1}{2}S_{10} = \frac{1}{2} - \left(\frac{1}{2}\right)^{11}$,

$$\text{and so } S_{10} = 2\left(\frac{1}{2} - \left(\frac{1}{2}\right)^{11}\right) = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

This procedure can be generalized to find the value S_n of any finite geometric series. Let S_n be the geometric series with first term g_1 , constant ratio $r \neq 1$, and n terms.

$$S_n = g_1 + g_1r + g_1r^2 + \dots + g_1r^{n-1}$$

$$rS_n = g_1r + g_1r^2 + \dots + g_1r^{n-1} + g_1r^n \quad \text{Multiply by } r.$$

$$S_n - rS_n = g_1 - g_1r^n$$

Subtract the second equation from the first.

$$(1 - r)S_n = g_1(1 - r^n)$$

Use the Distributive Property.

$$S_n = \frac{g_1(1 - r^n)}{1 - r}$$

Divide each side by $1 - r$.

This proves the following theorem.

Finite Geometric Series Formula

Let S_n be the sum of the first n terms of the geometric sequence with first term g_1 and constant ratio $r \neq 1$. Then $S_n = \frac{g_1(1 - r^n)}{1 - r}$.

The constant ratio r cannot be 1 in this formula. (Do you see why?) But that is not a problem. If $r = 1$, the series is $g_1 + g_1 + g_1 + \dots + g_1$, with n terms, and its sum is ng_1 .

STOP QY

The formula for a geometric series works even when the constant ratio is negative.

QY

Find the sum of the first 10 terms of a geometric series sequence with common ratio $\frac{3}{4}$ and first term 16.

GUIDED

Example 2

- Write the indicated sum given by $\sum_{k=1}^5 27\left(-\frac{1}{3}\right)^{k-1}$.
- Compute the value of the series in Part a.

Solution

- The indicated sum is $27(\underline{\quad})^0 + 27(\underline{\quad})^1 + 27(\underline{\quad})^{\underline{\quad}} + 27(\underline{\quad})^{\underline{\quad}} + \underline{\quad}(\underline{\quad})^{\underline{\quad}} = \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad}$.
- Use the Finite Geometric Series Formula.

$$S_5 = \frac{?(1 - ??)}{1 - ?} = \underline{\quad}$$

Check Compute the sum by hand to check that the formula works for negative values of r .

$$\underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \underline{\quad} = \underline{\quad}. \text{ It checks.}$$

Geometric Series and Compound Interest

Geometric series arise in compound-interest situations when the equal amounts of money are deposited or invested at regular intervals. The total value of such investments can be found using the Finite Geometric Series Formula.

Example 3

On the day her granddaughter Savanna was born, Mrs. Kash began saving for Savanna's college education by depositing \$1000 in an account earning an annual percentage yield of 5.2%. She continued to deposit \$1000 each year on Savanna's birthday into the same account at the same interest rate. How much money will be in Savanna's account on her 18th birthday, not including that birthday's payment?

Solution Make a table showing each deposit and its value on Savanna's 18th birthday.

The amount in the account is the value of the geometric series $1000(1.052) + 1000(1.052)^2 + \dots + 1000(1.052)^{18}$.

The first term is

$a = 1000(1.052)$, the ratio is

$r = 1.052$, and there are 18 terms. Therefore, the sum is

$$\frac{1000(1.052)(1 - 1.052^{18})}{1 - 1.052} \approx 30,153.58.$$

Savanna will have \$30,153.58 in the account on her 18th birthday.

Birthday	Deposit	Value on 18th Birthday
0	\$1000	$\$1000(1.052)^{18}$
1	\$1000	$\$1000(1.052)^{17}$
2	\$1000	$\$1000(1.052)^{16}$
\vdots	\vdots	\vdots
17	\$1000	$\$1000(1.052)^1$



Questions

COVERING THE IDEAS

- Refer to the Activity. What fraction of the square is shaded when $n = 8$?
- State a formula for the sum of the first n terms of a geometric sequence with first term g_1 and constant ratio r .
 - In the formula in Part a, what value can r not have?
 - Why can r not have the value in Part b? In this situation, what is the value of the series?

In 3–6, a geometric series is given.

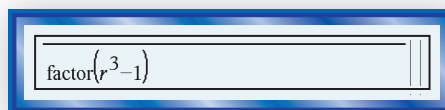
- How many terms does the series have?
 - Write the series in Σ -notation.
 - Use the Finite Geometric Series Formula to evaluate the series.
- $5 + 10 + 20 + 40 + \dots + 5 \cdot 2^7$
 - $170 + 17 + 1.7 + 0.17 + 0.017 + 0.0017$
 - $170 - 17 + 1.7 - 0.17 + 0.017 - 0.0017$
 - $a + \frac{1}{2}a + \frac{1}{4}a + \frac{1}{8}a + \dots + \left(\frac{1}{2}\right)^{16}a$
- Consider the geometric series in Example 2.
 - Calculate the following sums.
 - S_2
 - S_3
 - S_4
 - S_5
 - Plot the sums S_1 (which is 27), S_2 , S_3 , S_4 , and S_5 on a number line. How is S_n related to S_{n-1} and S_{n-2} ?
 - Find the sum of the first 17 terms of the geometric sequence with first term 20 and constant ratio 1.
 - Suppose \$500 is deposited into a bank account on July 17 for seven consecutive years and earns an annual percentage yield of 4%.
 - Write a geometric series that represents the value of this investment on July 17 of the eighth year (before that year's deposit).
 - Rewrite your answer to Part a in Σ -notation.
 - How much is in the account on July 17 of the eighth year?

APPLYING THE MATHEMATICS

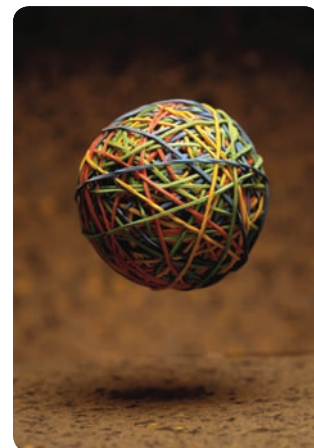
- A worker deposits \$2000 at the end of each year into a retirement account earning an annual percentage yield of 5.1%. Assume no other deposits or withdrawals from the account. To the nearest dollar, how much will the worker have after 40 years, assuming no deposit is made at the end of the 40th year?
 - A second worker waits ten years before starting to save money for retirement. Assume that this worker saves \$A per year for thirty years, also earning an APY of 5.1%. Write an expression for the total amount of money this worker will have saved, including interest, over thirty years, assuming no deposit is made at the end of the 30th year.
 - How much does the second worker have to save each year in order to have the same amount after 30 years as the first worker has after 40 years?

11. a. Complete the table below by expanding or factoring each polynomial on a CAS.

Standard Form	Factored Form
?	$(r - 1)(r + 1)$
$r^3 - 1$?
$r^4 - 1$?
$r^5 - 1$?
$r^6 - 1$?



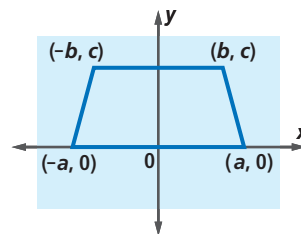
- b. **Fill in the Blanks** According to the Factor Theorem, $r - 1$ is a factor of a polynomial $P(r)$ if and only if $P(\underline{\quad}) = \underline{\quad}$. Use this fact to prove that $r - 1$ is a factor of $r^n - 1$ for all n .
- c. Use the results of Parts a and b to simplify $\frac{r^n - 1}{r - 1}$ for $n = 2, 3, 4, 5$, and 6 .
12. A ball is dropped from a height of 10 feet. Each bounce returns it to $\frac{4}{5}$ of the height of the previous bounce.
- Draw a diagram showing the ball's path until it touches the ground for the fourth time.
 - Find the total vertical distance the ball has traveled when it touches the ground for the fourth time.
13. Consider the geometric series $x^7 + x^{12} + x^{17} + \dots + x^{77}$.
- What is the common ratio?
 - Use the Finite Geometric Series Formula to find a formula for the value of the series.
 - Verify your result in Part b with a CAS.
14. The number $\frac{1}{3}$ can be approximated by the finite geometric series $0.3 + 0.03 + 0.003 + 0.0003 + \dots + g_1 r^{n-1}$.
- Identify g_1 and r for the series.
 - What approximation to $\frac{1}{3}$ occurs when $n = 6$?
 - How far from $\frac{1}{3}$ is the approximation in Part b?



REVIEW

15. Find the sum of the integers from 101 to 200. (Lesson 13-1)
16. a. How many odd integers are there from 25 to 75?
b. Find the sum of the odd integers from 25 to 75. (Lesson 13-1)
17. A math club mails a monthly newsletter. In January, the club mailed newsletters to each of its 325 members. If membership increases by 5 members each month, how many newsletters will it mail for the entire year? (Lesson 13-1)

18. Suppose $t_n = -2n + 9$. Find $t_1 + t_2 + t_3 + \dots + t_{19}$.
(Lessons 13-1, 3-8)
19. Let $f(x) = 27^x$. (Lessons 9-7, 7-7, 7-3, 7-2, 7-1)
- Evaluate $f(-3)$, $f(0)$, and $f\left(\frac{2}{3}\right)$.
 - Identify the domain and range of f .
 - Give an equation for the reflection image of the graph of $y = f(x)$ over the line $y = x$.
20. a. Identify the type of quadrilateral graphed at the right.
b. Prove or disprove that the diagonals of this quadrilateral have the same length. (Lesson 4-4, Previous Course)
21. Write an equation for the line parallel to $7x + 2y = 13$ and containing the point $(6, 5)$. (Lessons 3-4, 3-2)



EXPLORATION

22. Suppose that, because of inflation, a payment of P dollars n years from now is estimated to be worth $P(0.96)^n$ today. This worth is called the *present value* of a future payment.
- A lottery advertising \$10,000,000 in winnings actually plans to pay the winner \$500,000 each year for 20 years, starting the day the winner wins the lottery. Write the sum of the present values of these payments as a geometric series.
 - Evaluate the geometric series in Part a. How does the sum compare to the advertised jackpot of \$10,000,000?

QY ANSWERS

$$S_{10} = \frac{16\left(1 - \left(\frac{3}{4}\right)^{10}\right)}{1 - \frac{3}{4}} \approx$$

60.396