Lesson 13-1 Arithmetic Series

BIG IDEA There are several ways to find sums of the successive terms of an arithmetic sequence.

Vocabulary

series

arithmetic series

Σ, sigma

Σ-notation, sigma notation, summation notation

index variable, index

Sums of Consecutive Integers

There is a story the famous mathematician Carl Gauss often told about himself. When he was in third grade, his class misbehaved and the teacher gave the following problem as punishment:

"Add the whole numbers from 1 to 100."

Gauss solved the problem in almost no time at all. His idea was the following. Let S be the desired sum.

$$S = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$

Using the Commutative Property of Addition, the sum can be rewritten in reverse order.

$$S = 100 + 99 + 98 + \ldots + 3 + 2 + 1$$

Now add corresponding terms in the equations above. The sums 1 + 100, 2 + 99, 3 + 98, ... all have the same value!

So $\quad 2S = \underbrace{101 + 101 + 101 + \ldots + 101 + 101 + 101}_{100 \text{ terms}}.$

Thus, $\quad 2S = 100 \cdot 101$

and $\quad S = 5050.$

Gauss wrote only the number 5050 on his slate, having done all the figuring in his head. The teacher (who had hoped the problem would keep the students working for a long time) was quite irritated. However, partly as a result of this incident, the teacher did recognize that Gauss was extraordinary and gave him some advanced books to read. (You read about Gauss's work in Lesson 11-6 and may recall that he proved the Fundamental Theorem of Algebra at age 18.)

STOP **QY1**

Mental Math

Consider the arithmetic sequence defined by $a_n = 3n - 12$.

a. Find a_1, a_2, and a_3.

b. Find $a_1 + a_2 + a_3$.

c. Find a_{101}, a_{102}, and a_{103}.

d. Find $a_{101} + a_{102} + a_{103}$.

QY1

Use Gauss's method to add the integers from 1 to 40.

What Is an Arithmetic Series?

Recall that an *arithmetic* or *linear sequence* is a sequence in which the difference between consecutive terms is constant. An arithmetic sequence has the form

$$a_1, a_1 + d, a_1 + 2d, ..., a_1 + (n - 1)d, ... ,$$

where a_1 is the first term and d is the constant difference. For example, the odd integers from 1 to 999 form a finite arithmetic sequence with $a_1 = 1$, $n = 500$, and $d = 2$.

A **series** is an indicated sum of terms of a sequence. For example, for the sequence 1, 2, 3, a series is the indicated sum 1 + 2 + 3. The addends 1, 2, and 3 are the *terms* of the series. The value, or sum, of the series is 6. In general, the sum of the first n terms of a series a is

$$S_n = a_1 + a_2 + a_3 + ... + a_{n-2} + a_{n-1} + a_n.$$

If the terms of a series form an arithmetic sequence, the indicated sum of the terms is called an **arithmetic series**.

If a is an arithmetic series with first term a_1 and constant difference d, you can find a formula for the value S_n of the series by writing the series in two ways:

Start with the first term a_1 and successively add the common difference d.

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n - 1)d)$$

Start with the last term a_n and successively subtract the common difference d.

$$S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n - 1)d)$$

Now add corresponding pairs of terms of these two formulas, as Gauss did. Then each of the n pairs has the same sum, $a_1 + a_n$.

$$S_n + S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n)}_{n \text{ terms}}$$

So $\quad 2S_n = n(a_1 + a_n).$

Thus, $\quad S_n = \frac{n}{2}(a_1 + a_n).$

This proves that if $a_1 + a_2 + ... + a_n$ is an arithmetic series, then a formula for the value S_n of the series is $S_n = \frac{n}{2}(a_1 + a_n)$.

STOP QY2

Arithmetic series that involve the sum of consecutive integers from 1 to n lead to a special case of the above formula. In these situations, $a_1 = 1$ and $a_n = n$, so the sum of the integers from 1 to n is $\frac{n}{2}(1 + n)$, or $\frac{n^2 + n}{2}$.

▶ QY2

Use the formula for S_n to find the sum of the odd integers from 1 to 999.

GUIDED

Example 1

Part of the lyrics of a popular Christmas carol say, "On the 12th day of Christmas my true love gave to me… ." The 12th-day gifts are listed at the right. How many gifts did the singer receive on the 12th day of Christmas?

Solution Find the sum of consecutive integers from 1 to 12. Substitute 12 for n in the formula for S_n.

$$S_{12} = \frac{?}{2}(1 + \underline{\ ?\ })$$

$$S_{12} = \underline{\ ?\ }$$

The singer received $\underline{\ ?\ }$ gifts on the 12th day of Christmas.

The formula $S_n = \frac{n}{2}(a_1 + a_n)$ is convenient if the first and nth terms of the series are known. If the nth term is not known, you can use another formula. Start with the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n - 1)d$$

Substitute this expression for a_n in the right side of the formula for S_n and simplify.

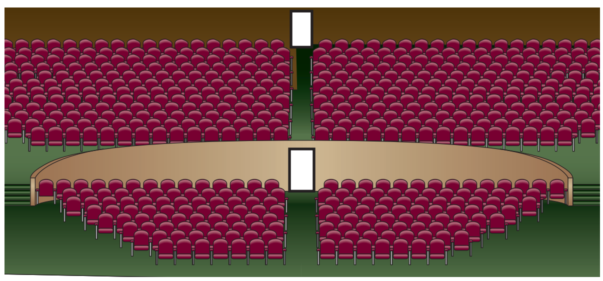
$$S_n = \frac{n}{2}\left[a_1 + (a_1 + (n - 1)d\right]$$

$$S_n = \frac{n}{2}\left[2a_1 + (n - 1)d\right]$$

This argument proves that if $a_1 + a_2 + \ldots + a_n$ is an arithmetic series with constant difference d, then the value S_n of the series can be found using the formula $S_n = \frac{n}{2}\left[2a_1 + (n - 1)d\right]$.

Example 2

An auditorium has 20 rows, with 14 seats in the front row and 2 more seats in each row thereafter. How many seats are there in all?



Solution This is an arithmetic-series situation. Because the first term, difference, and number of terms are given, use the formula

$$S_n = \frac{n}{2}(2a_1 + (n - 1)d).$$

In this case, $n = 20$, $a_1 = 14$, and $d = 2$.

So $$S_{20} = \frac{20}{2}(2 \cdot 14 + (20 - 1)2)$$

$$= \frac{20}{2}(28 + 38) = 660.$$

There are 660 seats in the auditorium.

Check Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. You need to know how many seats are in the first and last row. In this case,

$a_1 = 14$ and $a_n = a_{20} = 14 + 19 \cdot 2 = 52$.

So $S_{20} = \frac{20}{2}(14 + 52) = \frac{20}{2} \cdot 66 = 660$.

There are 660 seats in the auditorium. It checks.

Summation Notation

The sum of the first six terms of a sequence a_n is

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6.$$

However, when there are many numbers in the series, this notation is too cumbersome. You can shorten this by writing

$$a_1 + a_2 + \cdots + a_6.$$

It is understood that the terms a_3, a_4, and a_5 are included.

This notation can be shortened even further. In a spreadsheet, suppose you have the sum A1 + A2+ A3 + A4 + A5 + A6. That sum can be written as `SUM(A1:A6)`. In algebra, the upper-case Greek letter **Σ (sigma)** indicates a sum. In **Σ-notation,** called **sigma notation** or **summation notation,** the above sum is written

$$\sum_{i=1}^{6} a_i.$$

The expression can be read as "the sum of the values of a sub i, for i equals 1 to 6." The variable i under the Σ sign is called the **index variable,** or **index**. It is common to use the letters $i, j, k,$ or n as index variables. (In summation notation, i is *not* the complex number $\sqrt{-1}$.) In this book, index variables have only integer values.

READING MATH

While you use your index finger to point an object, the *index variable* is used to point to a value. Thus, a_3 points to the third term of the series named a.

Writing Formulas Using Σ-Notation

The two arithmetic series formulas $S_n = \frac{n}{2}(a_1 + a_n)$ and $S_n = \frac{n}{2}(2a_1 + (n - 1)d)$ can be restated using Σ-notation. Notice that i is used as the index variable to avoid confusion with the variable n.

Arithmetic Series Formula

In an arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$ with constant difference d,

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n - 1)d).$$

When $a_i = i$, the sequence is the set of all positive integers in increasing order 1, 2, 3, 4, … . Then

$$\sum_{i=1}^{n} i = \frac{n}{2}(1 + n) = \frac{n(n+1)}{2}.$$

This is a Σ-notation version of Gauss's sum.

STOP QY3

One advantage of Σ-notation is that you can substitute an expression for a_i. For instance, suppose $a_n = 2n$, the sequence of even positive integers. Then,

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} (2i) = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6$$
$$= 2 + 4 + 6 + 8 + 10 + 12$$
$$= 42.$$

The sum of the first six positive even integers is 42.

▶ QY3

Find $\sum_{i=1}^{40} i$.

GUIDED

Example 3

Consider $\sum_{i=1}^{500} a_i$, where $a_n = 4n + 6$.

a. Write the series without Σ-notation.

b. Evaluate the sum.

Solution

a. Substitute the expression for a_i from the explicit formula and use it to write out the terms of the series.

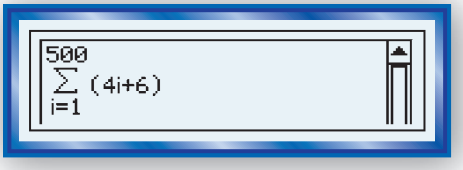
$$\sum_{i=1}^{500} (4i + 6) = (4 \cdot \underline{\ ?\ } + 6) + (4 \cdot \underline{\ ?\ } + \underline{\ ?\ }) + \underline{\ ?\ } + \ldots + \underline{\ ?\ }$$
$$= \underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ } + \ldots + \underline{\ ?\ }$$

b. This is an arithmetic series. The first term is ___?___. The constant difference is ___?___. There are ___?___ terms in the series.

Use the formula $\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1 + (n - 1)d)$ to evaluate the series.

$$\sum_{i=1}^{500} (4i + 6) = \frac{?}{2}(2 \cdot \underline{\ ?\ } + (\underline{\ ?\ } - 1)\underline{\ ?\ })$$
$$= \underline{\ ?\ }$$

Most scientific calculators and CAS have commands to evaluate a series, but the commands and entry styles vary considerably. The entry for one CAS is shown at the right.



STOP **QY4**

▶ QY4

Check the solution to Example 3 on your calculator or CAS.

Questions

COVERING THE IDEAS

In 1 and 2, tell whether what is given is an arithmetic sequence, an arithmetic series, or neither.

1. $26 + 29 + 32 + 35$

2. 35, 32, 29, 26

3. What problem was Gauss given in third grade, and what is its answer?

4. Find the sum of the integers from 1 to 500.

5. Fill in the Blank The symbol Σ is the upper-case Greek letter ___?___.

6. Fill in the Blank In Σ-notation, the variable under the Σ sign is called the ___?___.

7. Consider the arithmetic sequence with first term a_1 and constant difference d.

a. Write a formula for the nth term.

b. Write a formula for the sum S_n of the first n terms using Σ-notation.

c. Write an equivalent formula to the one you wrote in Part b.

8. Consider the arithmetic series $8 + 13 + 18 + ... + 38$.

a. Write out all the terms of the series. How many terms are there?

b. What is the sum of all the terms?

c. Write this series using Σ-notation.

9. Refer to Example 3. Check your answer using the formula $\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$.

10. Multiple Choice $\sum_{i=1}^{5} i^2 =$ ___?___.

A $1 + 4 + 9 + 16 + 25$ **B** 5^2

C $1 + 2 + ... + 5$ **D** none of these

11. Multiple Choice $7 + 14 + 21 + 28 + 35 + 42 + 49 + 56 + 63 = \underline{\ ?\ }$.

A $\sum_{i=7}^{63} i$ B $\sum_{i=7}^{63} (7i)$ C $\sum_{i=1}^{9} (7i)$ D none of these

12. In $\sum_{i=100}^{300} (5i)$, how many terms are added? (*Be careful!*)

In 13 and 14, evaluate the sum.

13. $\sum_{i=1}^{25} (6i - 4)$

14. $\sum_{n=-1}^{3} 9 \cdot 3^n$

APPLYING THE MATHEMATICS

15. Finish this sentence: The sum of the first n terms of an arithmetic sequence equals the average of the first and last terms multiplied by $\underline{\quad ? \quad}$.

16. The Jewish holiday Chanukah is celebrated by lighting candles in a *menorah* for eight nights. On the first night, two candles are lit, one in the center and one on the right. The two candles are allowed to burn down completely. On the second night, three candles are lit (one in the center, and two others) and are allowed to burn down completely. On each successive night, one more candle is lit than the night before, and all are allowed to burn down completely. How many candles are needed for all eight nights?

17. Penny Banks decides to start saving money in a Holiday Club account. At the beginning of January she will deposit \$100, and each month thereafter she will increase the deposit amount by \$25. How much will Penny deposit during the year?

18. a. How many even integers are there from 50 to 100?

b. Find the sum of the even integers from 50 to 100.

19. a. Write the sum of the squares of the integers from 1 to 100 in Σ-notation.

b. Evaluate the sum in Part a.

20. a. Translate this statement into an algebraic formula using Σ-notation: The sum of the cubes of the integers from 1 to n is the square of the sum of the integers from 1 to n.

b. Verify the statement in Part a when $n = 8$

21. Write the arithmetic mean of the n numbers $a_1, a_2, a_3, \ldots, a_n$ using Σ-notation.

REVIEW

22. The function with equation $y = k \cdot 2^x$ contains the point (6, 10).

a. What is the value of k?

b. Describe the graph of this function. **(Lesson 9-1)**

23. Consider the geometric sequence 16, –40, 100, –250, … .

a. Determine the common ratio.

b. Write the 5th term.

c. Write an explicit formula for the nth term. **(Lesson 7-5)**

24. Suppose an account pays 5.75% annual interest compounded monthly. **(Lesson 7-4)**

a. Find the annual percentage yield on the account.

b. Find the value of a $1700 deposit after 5 years if no other money is added or withdrawn from the account.

25. Find an equation for the parabola with a vertical line of symmetry that contains the points (5, 0), (1, 5), and (3, 8). **(Lesson 6-6)**

26. a. Find the inverse of $\begin{bmatrix} 5 & 2 \\ n & 1 \end{bmatrix}$.

b. For what value(s) of n does the inverse not exist? **(Lesson 5-5)**

EXPLORATION

27. The number 9 can be written as an arithmetic series: $9 = 1 + 3 + 5$. What other numbers from 1 to 100 can be written as an arithmetic series with three or more positive integer terms?

QY ANSWERS

1. $S = 1 + 2 + ... + 40$
$S = 40 + 39 + ... + 1$
$2S = \underbrace{41 + 41 + ... + 41}_{40 \text{ terms}}$
$2S = 40 \cdot 41$
$S = 820$

2. $S_n = \frac{500}{2}(1 + 999) = 250 \cdot 1000 = 250{,}000$

3. 820

4. Answers vary. Sample: