Lesson

9-10 Using Logarithms to Solve Exponential Equations

▶ **BIG IDEA** The solution to the exponential equation $b^x = a$ can be written as $\log_b a$, or it can be written as $\frac{\log a}{\log b}$.

Methods for Solving $b^x = a$

You already know a few ways to solve equations of the form $b^x = a$. When a is an integer power of b, you may solve the equation in your head. For instance, to solve $4^x = 64$, you may know that 3 is a solution, because $4^3 = 64$.

Sometimes you may notice that a and b are powers of the same number. For instance,

$$100^x = \sqrt{10}$$

can be solved by noting that both sides of the equation can be written as powers of 10.

$$100^x = (10^2)^x = 10^{2x} \quad \text{and} \quad \sqrt{10} = 10^{0.5}$$

Substitute into the given equation.

$$10^{2x} = 10^{0.5}$$

Equate the exponents and solve the resulting equation for x.

$$2x = 0.5$$

$$x = 0.25$$

Still another way to solve an equation of the form $b^x = a$ is to graph $y = b^x$ and see where the graph intersects the horizontal line $y = a$.

And, of course, you could use a CAS to solve $b^x = a$ directly.

But what happens if a and b are not easily found powers of the same number and a graphing utility or a CAS is not available? Then you can use logarithms to solve these equations.

Mental Math

Solve.

a. $3^x = 81$

b. $5^y + 75 = 200$

c. $4 \cdot 7^z = 196$

d. $2^{w+2} - 16 = 0$

Using Logarithms to Solve $b^x = a$

The process is similar to the process you have used to solve many equations where the unknown is on one side of the equation. Instead of adding the same number to both sides or multiplying both sides by the same number, you take the logarithm of both sides. Then apply the Logarithm of a Power Theorem to solve the resulting equation. When the base of the exponential equation is e, use natural logarithms.

Example 1

At what rate of interest, compounded continuously, do you have to invest your money so that it will double in 12 years?

Solution Use the Continuously Compounded Interest Formula $A = Pe^{rt}$. You want to know the rate at which after 12 years A will equal twice P, so substitute $2P$ for A and 12 for t.

$$A = 2P = Pe^{12r}$$

Since P is not zero, you can divide both sides by P.

$$2 = e^{12r}$$

Because the base in the equation is e, it is easiest to take the natural logarithm of each side.

$$\ln 2 = \ln(e^{12r})$$

$$\ln 2 = 12r \qquad \text{Log}_b \text{ of } b^n \text{ Theorem}$$

$$r = \frac{\ln 2}{12} \approx 0.0578 = 5.78\%$$

Check Choose a value for P and substitute the values back into the formula. We use $P = \$500$.

$A = 500 \cdot e^{(0.0578)(12)} \approx 1000$, or twice the investment. It checks.

STOP QY

QY

Why would it be less efficient to use common logarithms in Example 1?

Decay or depreciation problems modeled by exponential equations with negative exponents can be solved in a similar way.

GUIDED

Example 2

Archie Oligist finds the remains of an ancient wood cooking fire and determines that it has lost about 23.2% of the carbon-14 expected for that kind of wood. Recall that the half-life of carbon-14 is about 5730 years.

A radioactive decay model for this situation is $A = Ce^{-\frac{\ln 2}{5730}t}$, where C is the original amount of carbon-14 and A is the amount present after t years. Use the model to estimate the age of the wood.

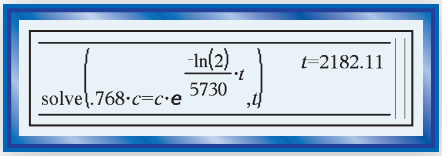
Solution Substitute values for the variables into the model. Because 23.2% of the carbon-14 has decayed, 76.8% remains and $A = 0.768C$.

$$A = Ce^{-\frac{\ln 2}{5730}t}$$

$0.768C = \underline{\ ?\ }$	Substitute for A.
$0.768 = \underline{\ ?\ }$	Divide both sides by C.
$\ln 0.768 = \underline{\ ?\ }$	Take the natural log of both sides.
$\ln 0.768 = \underline{\ ?\ }$	Log_b of b^n Theorem
$\underline{\ ?\ } \cdot \ln 0.768 = t$	Solve for t.
$t \approx \underline{\ ?\ }$	

The wood is about $\underline{\ ?\ }$ years old.

Check Solve on a CAS as shown below. It checks.



In the previous Examples, exponential equations were solved by taking natural logarithms of both sides because the base in the exponential equation was e. You can use any base of a logarithm to solve an exponential equation.

Example 3

Solve $7^x = 28$ by taking

a. the common logarithm of each side.

b. the natural logarithm of each side.

Solution Solutions for Parts a and b are written side-by-side. First read the solution to Part a (the columns at the left and center). Then read the solution to Part b (the center and right columns). Then reread both solutions by reading across each line.

a.		b.
$7^x = 2$		$7^x = 28$
$\log 7^x = \log 28$	Take the log of each side.	$\ln 7^x = \ln 28$
$x \log 7 = \log 28$	Logarithm of a Power Theorem	$x \ln 7 = \ln 28$
$x = \frac{\log 28}{\log 7}$	Divide both sides by the coefficient of x.	$x = \frac{\ln 28}{\ln 7}$
$x \approx 1.7124$	Evaluate with a calculator.	$x \approx 1.7124$

Logarithms to bases 10 and e are used in the solutions to Example 3. Any other base for the logarithms could have been used to solve the equation. Because the same results are *always* obtained regardless of the base, you may choose common logarithms, natural logarithms, or logarithms to any other base for a given situation.

Changing the Base of a Logarithm

While most graphing utilities have keys for the natural logarithm LN and common logarithm LOG, some of them do not have a single operation that allows you to find logarithms with bases other than 10 or e. However, with the following theorem you can convert logarithms with any base b, such as $\log_5 18$, to a ratio of either common logarithms or natural logarithms. The proof is a generalization of the process used in Examples 1–3.

Change of Base Theorem

For all positive real numbers a, b, and t, $b \neq 1$ and $t \neq 1$,

$$\log_b a = \frac{\log_t a}{\log_t b}.$$

Proof Suppose $\log_b a = x$.

Definition of logarithm	$b^x = a$
Take the log base t of each side.	$\log_t b^x = \log_t a$
Apply the Logarithm of a Power Theorem.	$x \log_t b = \log_t a$
Divide.	$x = \frac{\log_t a}{\log_t b}$
Transitive Property of Equality	$\log_b a = \frac{\log_t a}{\log_t b}$

The Change of Base Theorem says that the logarithm of a number to any base is the log of the number divided by the log of the base. Notice that the logarithms on the right side of the formula must have the same base.

Example 4

Approximate $\log_5 18$ to the nearest thousandth.

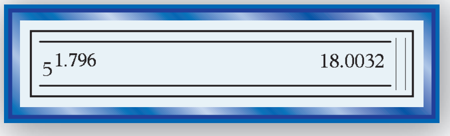
Solution 1 Use the Change of Base Theorem with common logarithms.

$$\log_5 18 = \frac{\log 18}{\log 5} \approx 1.796$$

Solution 2 Use the Change of Base Theorem with natural logarithms.

$$\log_5 18 = \frac{\ln 18}{\ln 5} \approx 1.796$$

Check By definition, $\log_5 18 \approx 1.796$ is equivalent to $5^{1.796} \approx 18$. The calculator display at the right shows that $5^{1.796} \approx 18$. It checks.



Questions

COVERING THE IDEAS

1. Solve $3^x = 243$
   a. in your head.
   b. by taking common logarithms.
   c. by taking natural logarithms.
2. Solve $20^r = 30$ to the nearest thousandth by using logarithms.

In 3 and 4, solve and check.

3. $\log_4 140 = y$
4. $53.75^z = 44$
5. Solve Example 1 using common logarithms.
6. Refer to Example 1. What interest rate, compounded continuously, would it take to triple your money in 12 years?
7. Refer to Example 2. Another artifact found several miles away contains only 48% of its expected carbon-14. About how old is this artifact?

In 8 and 9, approximate the logarithm to the nearest thousandth and check your answer.

8. $\log_2 30$
9. $\log_{0.5} 80$
10. Express as a single logarithm: $\frac{\log_8 P}{\log_8 Q}$.

APPLYING THE MATHEMATICS

11. Suppose you invest \$250 in a savings account paying 3.25% interest compounded continuously. How long would it take for your account to grow to \$300, assuming that no other deposits or withdrawals are made?
12. Suppose a colony of bacteria grows according to $N = Be^{2t}$, where N is the number of bacteria after t hours and B is the initial number in the colony. How long does it take the colony to grow to 10 times its original size?

In 13 and 14, solve.

13. $8^{4y} = 1492$
14. $\log_{12} 313 = -3x$

15. In 1999 the world population was estimated to be 5,995,000,000, and in 2000 the estimate was 6,071,000,000.

a. Calculate the percentage increase between 1999 and 2000.

b. Assume the percent increase remains the same. In what year will the world population reach 7 billion? Explain your answer.

c. In what year will the population be double the 2000 population?

d. Use the Internet to find the current and previous year's world population, and then calculate the percentage increase. With this growth rate, estimate when the world population will be twice the 2000 population.

16. In the formula $A = C\left(\frac{1}{2}\right)^{\frac{t}{H}}$, A is the amount of carbon left after time t, where C is the initial amount and H is the half-life. In this lesson the formula $A = Ce^{-\frac{\ln 2}{H}t}$ is used. Use the properties of exponents and logarithms to prove that $C\left(\frac{1}{2}\right)^{\frac{t}{H}} = Ce^{-\frac{\ln 2}{H}t}$ for all t.

In 17–19, between which two consecutive whole numbers does each logarithm fall? Answer without using a calculator.

17. $\log_3 21$ **18.** $\log_{12} 200$ **19.** $\log_7\left(\frac{1}{50}\right)$

REVIEW

In 20 and 21, solve and check. (Lesson 9-9)

20. $\log w = \frac{3}{2}\log 16 - \log 8$ **21.** $\log 125 = x \log 5$

22. Find the error in the following proof that $3 < 2$. (*Hint:* What is the sign of $\log\left(\frac{1}{3}\right)$?) (Lessons 9-9, 9-5, 5-1)

$$\frac{1}{27} < \frac{1}{9}$$
$$\log\left(\frac{1}{27}\right) < \log\left(\frac{1}{9}\right)$$
$$\log\left[\left(\frac{1}{3}\right)^3\right] < \log\left[\left(\frac{1}{3}\right)^2\right]$$
$$3\log\left(\frac{1}{3}\right) < 2\log\left(\frac{1}{3}\right)$$
$$3 < 2$$

23. For a 3-stage model rocket, the formula $V = c_1 \cdot \ln R_1 + c_2 \cdot \ln R_2 + c_3 \cdot \ln R_3$ is used to find the velocity of the rocket at final burnout. If $R_1 = 1.37$, $R_2 = 1.59$, $R_3 = 1.81$, $c_1 = 2185\ \frac{\text{m}}{\text{sec}}$, $c_2 = 2530\ \frac{\text{m}}{\text{sec}}$, and $c_3 = 2610\ \frac{\text{m}}{\text{sec}}$, find V. (Lesson 9-8)

24. Recall that the relative intensity of sound is given by $D = 10\log\left(\frac{N}{10^{-12}}\right)$. Find the decibel level of a sound that has an absolute intensity of $3.7 \cdot 10^{-5}$. **(Lesson 9-6)**

In 25 and 26, evaluate, given $g(x) = \log x$. (Lessons 9-5, 8-3)

25. $g(0.01)$

26. $g^{-1}(4)$

27. a. Graph the triangle represented by $\begin{bmatrix} -5 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. **(Lessons 4-5, 4-1)**

b. What kind of triangle is this?

c. What matrix describes the image of the triangle in Part a under the transformation given by $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$?

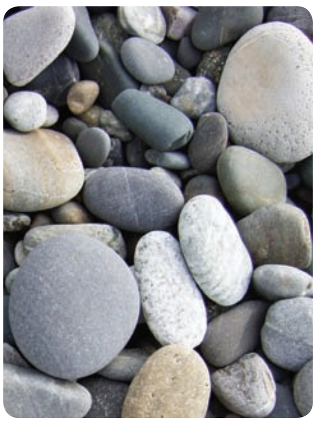
d. Graph the image on the same set of axes as in Part a.

e. Are the triangles in Parts a and c similar?

EXPLORATION

28. The table shows part of the Krumbein *phi* (ϕ) scale. Geologists use this scale to measure the grain size of the individual particles that make up rocks, soils, and other solids. Notice that the sizes in the first column are all powers of 2. Determine a logarithmic relationship between the size range and the ϕ-scale numbers in the second column.

Size Range (mm)	ϕ Scale	Aggregate Name	Other Names
> 256	< -8	Boulder	
64–256	–6 to –8	Cobble	
32–64	–5 to –6	Very coarse gravel	Pebble
16–32	–4 to –5	Coarse gravel	Pebble
8–16	–3 to –4	Medium gravel	Pebble
4–8	–2 to –3	Fine gravel	Pebble
2–4	–1 to –2	Very fine gravel	Granule
1–2	0 to –1	Very coarse sand	
$\frac{1}{2}$–1	1 to 0	Coarse sand	
$\frac{1}{4}$–$\frac{1}{2}$	2 to 1	Medium sand	



QY ANSWER

Answers vary. Sample: You would be required to solve $\log 2 = 12r \log e$.