Lesson 9-9 Properties of Logarithms

BIG IDEA Each basic property of powers corresponds to a basic property of logarithms.

The following properties of powers should be familiar both from Chapter 7 and from your work in previous courses. For all positive numbers x and for all real numbers m and n:

Zero Exponent Theorem	$x^0 = 1$
Product of Powers Postulate	$x^m \cdot x^n = x^{m+n}$
Quotient of Powers Theorem	$\frac{x^m}{x^n} = x^{m-n}$
Power of a Power Postulate	$(x^m)^n = x^{mn}$

Because every logarithm in base b is the exponent n of b^n, properties of logarithms can be derived from the properties of powers. In this lesson you will see five theorems, each related to one of the four properties of powers mentioned above.

Mental Math

Put in order from least to greatest.

a. $2 \cdot 10$, 2^{10}, 10^2

b. 3^{16}, π^{16}, e^{16}

c. 0.5, $\sqrt{0.5}$, $\sqrt[3]{0.5}$

The Logarithm of a Power of the Base

Recall that the base b of a logarithm can be any positive number other than 1.

Activity 1

Step 1 Evaluate each of the following:

a.	b.	c.
$\log_2 1$	$\log_2 2$	$\log_2 2^{-2}$
$\log_3 1$	$\log_3 3$	$\log_2 2^{-1}$
$\log_4 1$	$\log_4 4$	$\log_2 2^0$
$\ln 1$	$\ln e$	$\log_2 2^1$

Step 2 Describe the pattern or write a generalization for each of the three sets of expressions in Step 1.

Steps 1b and 1c of Activity 1 illustrate the following theorem. You already used this theorem in Lessons 9-7 and 9-8 to evaluate logarithms of numbers with various bases.

Log_b of b^n Theorem

For every positive base $b \neq 1$, and any real number n, $\log_b b^n = n$.

Proof

1. Suppose $\log_b b^n = x$.
2. $b^x = b^n$ Definition of logarithm
3. $x = n$ Equate the exponents.
4. $\log_b b^n = n$ Substitution (Steps 1 and 3)

So, if a number can be written as the power of the base, the exponent of the power is its logarithm.

Let $n = 0$ in the Log_b of b^n Theorem. Then $\log_b b^0 = 0$, and because $b^0 = 1$ for $b \neq 0$, $\log_b 1 = 0$. This should agree with what you found in Step 1a of Activity 1.

Logarithm of 1 Theorem

For every positive base $b \neq 1$, $\log_b 1 = 0$.

The Logarithm of 1 Theorem is a special case of the Log_b of b^n Theorem.

The Logarithm of a Product

The Product of Powers Postulate says that in order to multiply two powers with the same base, add their exponents. In particular, for any base b (with $b > 0$, $b \neq 1$) and any real numbers m and n, $b^m \cdot b^n = b^{m+n}$.

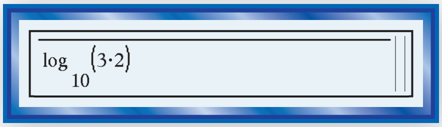
The corresponding property of logarithms is about the logarithm of a product of two numbers.

Activity 2

MATERIALS CAS

Step 1 Make a table like the one on the next page. Use a CAS to evaluate each expression, and then record the result.

A	$\log(3 \cdot 2)$	?
	$\log 3 + \log 2$	?
B	$\log\left(\frac{1}{6} \cdot 216\right)$	?
	$\log \frac{1}{6} + \log 216$	?
C	$\log_5\left(7 \cdot \frac{25}{2}\right)$	?
	$\log_5 7 + \log_5 \frac{25}{2}$	?
D	$\ln(4 \cdot 32)$	?
	$\ln 4 + \ln 32$	?



Step 2 Make a conjecture based on the results of Step 1:
$\log_b(x \cdot y) = \underline{\ ?\ }$.

Activity 2 shows that the logarithm of a product equals the sum of the logarithms of the factors. We state this result as a theorem and prove it using the Product of Powers Postulate.

Logarithm of a Product Theorem

For any positive base $b \neq 1$ and positive real numbers x and y, $\log_b(xy) = \log_b x + \log_b y$.

Proof Let $x = b^m$ and $y = b^n$, for any $b > 0$, $b \neq 1$. Let $z = xy$.

Then $\log_b x = m$ and $\log_b y = n$.	Definition of logarithm
So $\log_b(xy) = \log_b(b^m \cdot b^n)$	Substitution (Given)
$= \log_b(b^{m+n})$	Product of Powers Postulate
$= m + n$	Log_b of b^n Theorem
$= \log_b x + \log_b y$	Substitution (Step 1)

Example 1

Find $\log_8 128 + \log_8 4$ and check your result.

Solution By the Logarithm of a Product Theorem,

$\log_8 128 + \log_8 4 = \log_8(128 \cdot 4) = \log_8 512$

Because $512 = 8^3$, $\log_8 512 = 3$.

So, $\log_8 128 + \log_8 4 = 3$.

Check Notice that 8, 128, and 4 are all integer powers of 2.

Let $x = \log_8 128$. Then $8^x = 128$, or $2^{3x} = 2^7$. So, $x = \frac{7}{3}$.

Let $y = \log_8 4$. Then $8^y = 4$, or $2^{3y} = 2^2$. So, $y = \frac{2}{3}$.

Consequently, $x + y = \frac{7}{3} + \frac{2}{3} = \frac{9}{3} = 3$. It checks.

Example 2

Write $\log_7 96$ as the sum of two logarithms.

Solution Find two positive integers whose product is 96.

$\log_7 96 = \log_7(3 \cdot 32)$

$= \log_7 3 + \log_7 32$ Logarithm of a Product Theorem

STOP QY

The Logarithm of a Quotient

A Logarithm of a Quotient Theorem follows from the related Quotient of Powers Theorem, $b^m \div b^n = b^{m-n}$.

Logarithm of a Quotient Theorem

For any positive base $b \neq 1$ and for any positive real numbers x and y,

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

The proof of the Logarithm of a Quotient Theorem is similar to that of the Logarithm of a Product Theorem. You are asked to complete the proof in Question 22.

QY

Write $\log_7 96$ as the sum of a different pair of logarithms than those used in Example 2.

GUIDED

Example 3

Recall from Lesson 9-6 that the formula $D = 10 \log\left(\frac{N}{10^{-12}}\right)$ is used to compute the number D of decibels from the sound intensity N measured in watts per square meter. Use the Logarithm of a Quotient Theorem to rewrite this formula without a quotient.

Solution By the Logarithm of a Quotient Theorem,

$\log\left(\frac{N}{10^{-12}}\right) = \log \underline{\ ?\ } - \log \underline{\ ?\ }$

$= \log \underline{\ ?\ } + \underline{\ ?\ }$, because $\log_{10}(10^x) = x$.

Thus, $D = 10 \log\left(\frac{N}{10^{-12}}\right)$ is equivalent to

$D = 10(\log \underline{\ ?\ } + \underline{\ ?\ })$.

Members of marching bands are encouraged to wear earplugs to prevent hearing loss.

The Logarithm of a Power

Recall that $\log_b b^n = n$. In Activity 3 you will explore $\log_b x^n$, where $x \neq b$.

Activity 3

Step 1 Estimate each logarithm to the nearest thousandth.

a. $\log 3$ **b.** $\log 3^2$
c. $\log 3^3$ **d.** $\log 3^4$
e. $\ln 10$ **f.** $\ln 100$
g. $\ln 1000$ **h.** $\ln 10{,}000$

Step 2 Make a conjecture based on the results of Step 1: $\log_b x^n =$ __?__.

Step 3 In Step 1 you should have found that $\log 3^4 = \log 3 + \log 3 + \log 3 + \log 3 = 4 \log 3$. What property of logarithms supports this conclusion?

Activity 3 illustrates the following theorem.

Logarithm of a Power Theorem

For any positive base $b \neq 1$ and for any positive real number x and any real number n, $\log_b(x^n) = n \log_b x$.

You are asked to complete the proof of this theorem in Question 23.

You can use the properties of logarithms to solve equations.

Example 4

Solve for k: $\log k = 2 \log 5 + \log 6 - \log 3$.

Solution Use the properties of logarithms.

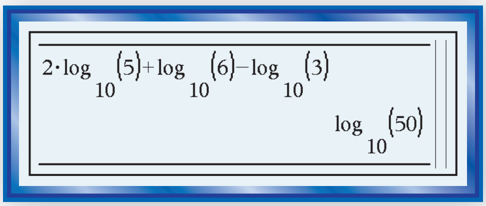
$\log k = \log 5^2 + \log 6 - \log 3$ Logarithm of a Power Theorem

$\log k = \log(5^2 \cdot 6) - \log 3$ Logarithm of a Product Theorem

$\log k = \log\left(\frac{5^2 \cdot 6}{3}\right) = \log 50$ Logarithm of a Quotient Theorem, arithmetic

$k = 50$

Check Evaluate $2 \log 5 + \log 6 - \log 3$ on a CAS. The result, as shown at the right, is $\log 50$. It checks.



Questions

COVERING THE IDEAS

In 1–3, simplify.

1. $\log_{5.1} 5.1^{15.4}$
2. $\log_r r^n$
3. $\log_p 1$
4. Write $\log_7 96$ as the sum of a pair of logarithms different from the pair used in Example 2 and the QY.

In 5–7, an expression is given.

a. **Write it as the logarithm of a single number or as a number without a logarithm.**

b. **Name the property or properties of logarithms that you used.**

5. $\log_2 49 - \log_2 7$
6. $\log_{64} 3 + \log_{64} 8 - \log_{64} 4$
7. $3 \log_3 \sqrt{9}$
8. Here is the start of a proof to show that $\log x^4 = 4 \log(x)$ without using the Logarithm of a Power Theorem. Begin by realizing $\log x^4 = \log(x \cdot x \cdot x \cdot x)$. Next, use the Logarithm of a Product Theorem. With this hint, complete the rest of the proof.
9. A student enters $\log(64)$ on a CAS and gets $6 \log 2$ as an output. Show that the two expressions are equivalent using the properties of logarithms.

In 10 and 11, rewrite the expression as a single logarithm.

10. $\log 14 - 3 \log 4$
11. $\log_b x - \log_b y + \frac{1}{2} \log_b z$

True or False In 12–17, if the statement is false, give a counterexample, and then correct the statement to make it true.

12. $\log_b(3x) = 3 \log_b x$
13. $\ln M - \ln N = \ln\left(\frac{M}{N}\right)$
14. $\frac{\log p}{\log q} = \log\left(\frac{p}{q}\right)$
15. $\log(M + N) = \log M + \log N$
16. $\log_7\left(\frac{M}{N}\right) = \frac{\log_7 M}{\log_7 N}$
17. $6 \log_2 T = \log_2(T^6)$

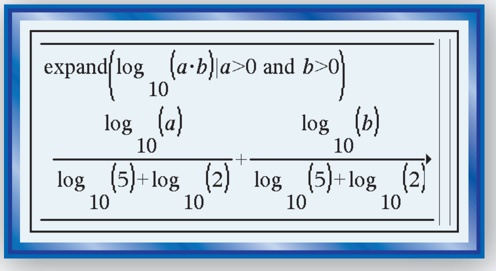
APPLYING THE MATHEMATICS

In 18 and 19, solve for *y*.

18. $\ln h = \frac{1}{4} \ln x + \ln y$
19. $\log w = \log\left(\frac{x}{y^4}\right)$

20. In the formula for the decibel scale $D = 10\log\left(\frac{N}{10^{-12}}\right)$, where N is the sound intensity and D is relative intensity, show that $D = 120 + \log N^{10}$ by using properties of logarithms.

21. Janella used a CAS to expand `log(a*b)`, where a and b are both positive. The display at the right shows her CAS output. Show that this CAS output is equivalent to the original expression.



22. **Fill in the Blanks** Justify each step in the following proof of the Logarithm of a Quotient Theorem.

Let $x = b^m$, $y = b^n$, and $z = \frac{x}{y}$.

a.	$\log_b x = m$ and $\log_b y = n$	a. ___?___
b.	$\log_b\left(\frac{x}{y}\right) = \log_b\left(\frac{b^m}{b^n}\right)$	b. ___?___
c.	$= \log_b($___?___$)$	c. ___?___
d.	$= m - n$	d. ___?___
e.	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	e. ___?___

23. **Fill in the Blanks** Complete the following proof of the Logarithm of a Power Theorem.

Let $\log_b x = m$.

a.	Then, $x =$ ___?___.	definition of logarithm
b.	$x^n =$ ___?___	Raise both sides to the nth power.
c.	$x^n =$ ___?___	Power of a Power Postulate
d.	$x^n =$ ___?___	Commutative Property of Multiplication
e.	$\log_b x^n =$ ___?___	definition of logarithm
	$\log_b x^n = n \log_b x$	substitution

REVIEW

24. Simplify without a calculator. **(Lesson 9-7)**

a. $\log_{81} 81$ b. $\log_{81} 9$ c. $\log_{81} 3$

d. $\log_{81} 1$ e. $\log_{81}\left(\frac{1}{81}\right)$ f. $\log_{81}\left(\frac{1}{9}\right)$

In 25 and 26, solve. (Lessons 9-7, 9-5)

25. $\log_x 144 = 2$

26. a. $\log y = -4$ b. $\log(-4) = z$

27. After World War I, harsh reparation payments imposed on Germany caused the value of the deutschmark to decline against foreign currencies and German wholesale prices to increase rapidly. The table at the right shows the wholesale price index in Germany from 1914 to 1922. **(Lesson 9-4)**

a. Sketch a scatterplot of these data.

b. Why do these data suggest an exponential model?

c. Find an exponential equation that models these data for semiannual periods after July 1914.

d. Use Part c to predict the wholesale price index for January 1923.

e. The actual wholesale price index for January 1923 was 2785. Does this model over or underpredict the actual wholesale price index?

Date	Wholesale Price Index
July 1914	1.0
Jan. 1919	2.6
July 1919	3.4
Jan. 1920	12.6
Jan. 1921	14.4
July 1921	14.3
Jan. 1922	36.7
July 1922	100.6

28. A student borrows \$2000 to go to college. The student loan accrues interest continously at a rate of 3.75% for four years. What is the total loan balance after four years? **(Lesson 9-3)**

29. a. Find the average rate of change between $x = -3$ and $x = -2$ for the function with equation $y = \frac{1}{3}x^3$. **(Lesson 7-1)**

b. Sketch a graph and use it to explain your answer to Part a.

30. **Multiple Choice** A distributor sells old-time movies from the 1940s and 1950s for \$5.99 each plus a shipping and handling charge of \$3.89 per order. As a bonus, every fifth movie ordered is free. Which equation gives the correct charge $f(m)$ for m movies? **(Lesson 3-9)**

A $f(m) = 3.89 + 5.99m - 5.99\left\lfloor \frac{m}{5} \right\rfloor$

B $f(m) = 3.89 + 5.99m - 5.99\left\lfloor 1 - \frac{m}{5} \right\rfloor$

C $f(m) = \frac{5.99m}{5} + 3.89$

D $f(m) = 3.89 + 5.99m - \frac{5.99m}{5}$

EXPLORATION

31. What common logarithms of whole numbers from 1 through 20 can you estimate without a calculator using only log 1, log 2 ≈ 0.301, log 3 ≈ 0.477, and log 10?

QY ANSWER

Answers vary. Sample:
$\log_7 12 + \log_7 8$