Lesson 9-8 Natural Logarithms

Vocabulary

natural logarithm of *m*, ln *m*

BIG IDEA When a number is written as a power of the irrational number *e*, the exponent in the expression is the natural logarithm of the number.

What Are Natural Logarithms?

Any positive number except 1 can be the base of a logarithm. The number *e* that you studied in Lesson 9-3 is frequently used as a logarithm base in real-world applications.

Logarithms to the base *e* are called ***natural logarithms***. Just as $\log x$ (without any base named) is shorthand for $\log_{10} x$, $\ln x$ is shorthand for $\log_e x$.

Definition of Natural Logarithm of *m*

n is the **natural logarithm of *m***, written $n = \ln m$, if and only if $m = e^n$.

The symbol $\ln x$ is usually read "natural log of *x*".

Natural logarithms of powers of *e* can be determined in your head from the definition.

$\ln 1 = \log_e 1 = 0$ because $1 = e^0$.

$\ln e = \log_e e = 1$ because $e = e^1$.
That is, $\ln 2.718 \approx 1$ because $e^1 \approx 2.718$.

$\ln e^3 = \log_e e^3 = 3$ because $e^3 = e^3$.
That is, $\ln 20.086 \approx 3$ because $e^3 \approx 20.086$.

In general, $\ln(e^x) = x$.

STOP **QY1**

The Wang Model 360E calculator, first developed in 1964, was one of the first calculators capable of computing logarithms.

Mental Math

Find an expression for $f \circ g(x)$ if

a. $f(x) = x^2$ and $g(x) = x^4$

b. $f(x) = x^4$ and $g(x) = x^2$

c. $f(x) = \sqrt{x}$ and $g(x) = 6x - 3$

d. $f(x) = 6x - 3$ and $g(x) = \sqrt{x}$

QY1

What is the value of $\ln\left(\frac{1}{e}\right)$?

Evaluating Natural Logarithms

Example 1

Estimate the following to the nearest thousandth using a calculator.

a. ln 100 b. ln 5

Solution Enter as shown at the right.

a. $\ln 100 \approx 4.605$

b. $\ln 5 \approx 1.609$

STOP QY2

> **QY2**
> Check your answers to Example 1 using powers of e.

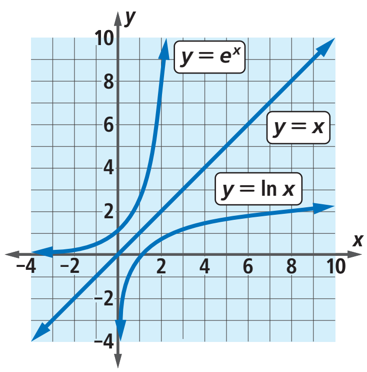
Caution! In many computer languages, the natural logarithm function is denoted as log(x). This can be confusing because in most other places, including on calculators, log(x) means the common log, with base 10.

The Graph of $y = \ln x$

The function with equation $y = \ln x$ is the inverse of the function with equation $y = e^x$, just as $y = \log x$ is the inverse of $y = 10^x$ and $y = \log_2 x$ is the inverse of $y = 2^x$. The inverse relationship of $y = e^x$ and $y = \ln x$ is displayed in the tables and graphs below. The graph of each function is the reflection image of the other over the line with equation $y = x$.

x	$y = e^x$
–1	0.37
0	1.00
1	$e \approx 2.72$
1.6	4.95

x	$y = \ln x$
0.37	–1
1.00	0
$e \approx 2.72$	1
4.95	1.6



The function f with equation $f(x) = \ln x$ has all the properties held by all other logarithmic functions. In particular, the domain of the natural logarithm function is the set of positive real numbers, and its range is the set of all real numbers.

Applications of Natural Logarithms

Natural logarithms are frequently used in formulas.

GUIDED

Example 2

According to the Beer-Lambert law, if you shine a 10-lumen light into a lake, the light intensity I (in lumens) at a depth of d feet under water is given by $d = -k \cdot \ln\left(\frac{I}{10}\right)$, where k is a measure of the light absorbance of the water. For a lake where $k = 34$, at what depth is the light intensity 6 lumens?

Solution Substitute the known values of k and I and solve for d.

$k = 34$ and $I =$ __?__

So, $d =$ __?__ $\cdot \ln\left(\frac{?}{10}\right)$

$d \approx$ __?__

The light intensity is 6 lumens at about __?__ feet.

Example 3

Let O_1 and O_2 be the temperatures of a cooling object before and after taking t minutes to cool, and let S_1 and S_2 be the temperatures of the surrounding environment before and after those t minutes. Newton's Law of Cooling says that t varies directly as the natural log of the ratio of the differences in temperature between the object and the surrounding area, or $t = k \ln\left(\frac{O_2 - S_2}{O_1 - S_1}\right)$.

The constant of variation k depends on the temperature scale (Celsius or Fahrenheit), the type of container the object is in, the altitude, and other environmental conditions.

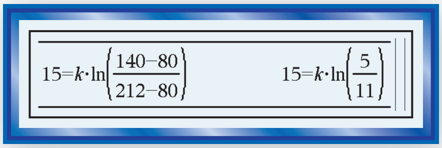
a. After bringing it to a boil (212°F), Laura let her soup cool for 15 minutes to 140°F. Use the data to find a value of k if Laura is making her soup in an 80°F kitchen.

b. Suppose Laura heats a pot of vegetable soup to boiling, and lets it cool 15 minutes before serving. What is the serving temperature of the soup if the kitchen is 70°F?

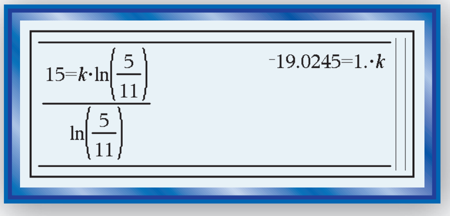
(continued on next page)

Solution

a. Substitute values for O_1, O_2, S_1, S_2, and t into the equation to solve for k. Note that because the kitchen temperatures did not change, $S_1 = S_2 = 80$. Enter the equation into a CAS as shown below. This CAS simplifies the fraction automatically.



Divide both sides by $\ln\left(\frac{5}{11}\right)$, as shown at the right.



So, $k \approx -19$.

b. From Part a, $T = -19 \ln\left(\frac{O_2 - S_2}{O_1 - S_1}\right)$. Substitute values for O_1, S_1, S_2, and t into the equation from Part a and divide both sides by -19.

$$15 = -19 \ln\left(\frac{O_2 - 70}{212 - 70}\right)$$

$$-\frac{15}{19} = \ln\left(\frac{O_2 - 70}{142}\right)$$

Use the definition of natural logarithm to rewrite the equation.

$$e^{-\frac{15}{19}} = \frac{O_2 - 70}{142}$$

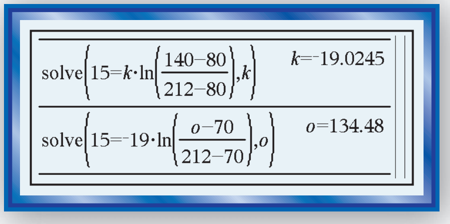
Calculate $e^{-\frac{15}{19}}$ and multiply both sides by 142.

$$0.454(142) \approx O_2 - 70$$

$$134 \approx O_2$$

The serving temperature of the soup is about 134°F.

Check Solve on a CAS as shown at the right.



Questions

COVERING THE IDEAS

1. What are logarithms to the base e called?

2. **Multiple Choice** Which of the following is not equivalent to $y = \ln x$? There may be more than one correct answer.

   A $x = \log_e y$ B $e^y = x$ C $y = \log_e x$ D $e^x = y$

In 3 and 4, write an equivalent exponential equation.

3. $\ln 1 = 0$

4. $\ln 1000 \approx 6.908$

In 5 and 6, write in logarithmic form.

5. $e^5 \approx 148.41$

6. $e^{0.5} \approx 1.65$

7. Without using a calculator, tell which is greater, ln 1000 or log 1000. How do you know?

8. Approximate ln 210 to the nearest thousandth.

9. Refer to Example 2. If the light intensity is 4.5 lumens, at what depth is the light in the lake?

10. Does the function with equation $f(x) = \ln x$ have an inverse? If so, what is it?

In 11–13, evaluate without using a calculator.

11. $\ln e$

12. $\ln e^4$

13. $\ln e^{-2}$

14. Refer to Part b of Example 3. What will the temperature of Laura's soup be if it cools for 10 minutes instead of 15?

APPLYING THE MATHEMATICS

15. Refer to Example 3. When cooling hot food in a refrigerator or freezer for service at a later time, caterers want to move through the temperature range from 145°F to 45°F as quickly as possible to avoid pathogen growth. How much longer will it take soup to go from 145° to 45° in a 34°F refrigerator than a 5°F freezer? (Assume $k = -19$.)

16. a. What happens when you try to find $\ln(-5)$ on a calculator?

b. Use the graph of $y = \ln(x)$ to explain why the calculator displayed what you saw in Part a.

17. In Lesson 9-6 you saw that the decibel scale is a logarithmic scale used to measure relative power intensity, often for sound. Named after John Napier, the *neper* is another measure of relative power intensity based on the natural log scale. A conversion formula between nepers and decibels is 1 neper $= \frac{20}{\ln 10}$ decibels. Find the number of decibels that are equivalent to 3 nepers.

18. How can you use a graph of $y = e^x$ to find the value of $\ln\left(\frac{2}{3}\right)$?

19. a. What is the y-intercept of the graph of $y = \ln x$?

b. What is the y-intercept of the graph of $y = \log x$?

c. Prove a generalization about the y-intercepts of the graphs of all equations of the form $y = \log_b x$.

In 20 and 21, suppose ln x = 9, ln y = 3, and ln z = 27. Evaluate.

20. $\ln(xyz)$

21. $\ln \sqrt[y]{\frac{z}{x}}$

REVIEW

22. Let $y = 7^x$. Write an equivalent logarithmic equation. **(Lesson 9-7)**

23. Solve for x: $\log_x 37 = 2$. **(Lesson 9-7)**

24. Use the earthquake moment magnitude scale formula $M_w = \log \frac{M_0}{1.5} - 10.7$. Determine the seismic moment M_0 for the earthquake in Chile, May 22, 1960, where $M_w = 9.5$, the largest (as of January 2007) ever recorded with a seismometer. **(Lesson 9-6)**

25. Multiple Choice Pick the equation for an exponential decay function. Explain your choice. **(Lesson 9-2)**

A $f(x) = \frac{1^x}{7}$ B $f(x) = 7^{-x}$ C $f(x) = 7^x$ D $f(x) = \left(\frac{1}{7}\right)^{-x}$

26. A 1:1000 scale model of the Empire State Building is a little less than $1\frac{1}{2}$ feet tall with a volume of 0.037 cubic foot. From this information, about how many cubic feet does the Empire State Building contain? **(Lesson 2-1)**

EXPLORATION

27. John Napier, the Scottish mathematician who invented logarithms, also invented a calculating device known as *Napier's bones*. Investigate how this device works. Use your findings to simulate and describe Napier's process for any 3-digit-by-2-digit multiplication.

Napier's bones was an early counting device.

QY ANSWERS

1. –1

2a. $e^{4.605} \approx 100$

b. $e^{1.609} \approx 5$