# Lesson 9-7 Logarithms to Bases Other Than 10

## Vocabulary

logarithm of *a* to the base *b*

logarithm function with base *b*

**BIG IDEA** When a number is written as a power of *b*, the exponent in the expression is the *logarithm of the number to the base b*.

In Lesson 9-5, you learned about the common logarithm, which allows you to solve for $x$ when $10^x = a$. But how do you solve $b^x = a$ when $b$ is any positive value other than 1? The same question was considered by Leonhard Euler in the 18th century. It led him to the following definition of a logarithm to any positive base other than 1.

### Mental Math

**Let $P = (3, 5)$. Give the coordinates of the image of $P$ under the given transformation.**

**a.** $r_y$

**b.** $S_3$

**c.** $S_{-2,-1}$

**d.** $T_{-2,-1}$

### Definition of Logarithm of *a* to the Base *b*

Let $b > 0$ and $b \neq 1$. Then $x$ is the **logarithm of *a* to the base *b***, written $x = \log_b a$, if and only if $b^x = a$.

For example, because $3^5 = 243$, you can write $5 = \log_3 243$. This is read "5 is the logarithm of 243 with base 3" or "5 is log 243 to the base 3" or "5 is the log base 3 of 243." At the right are some other powers of 3 and the related logs to the base 3.

| Exponential Form | Logarithmic Form |
|---|---|
| $3^4 = 81$ | $\log_3 81 = 4$ |
| $3^3 = 27$ | $\log_3 27 = 3$ |
| $3^2 = 9$ | $\log_3 9 = 2$ |
| $3^1 = 3$ | $\log_3 3 = 1$ |
| $3^{0.5} = \sqrt{3}$ | $\log_3 \sqrt{3} = 0.5$ |
| $3^0 = 1$ | $\log_3 1 = 0$ |
| $3^{-1} = \frac{1}{3}$ | $\log_3\left(\frac{1}{3}\right) = -1$ |
| $3^{-2} = \frac{1}{9}$ | $\log_3\left(\frac{1}{9}\right) = -2$ |
| $3^y = x$ | $\log_3 x = y$ |

STOP QY1

### QY1

Rewrite $7^4 = 2401$ in logarithmic form.

## Example 1

**Write the equation $P = 9(1.028)^x$ in logarithmic form.**

**Solution** First rewrite the equation in $b^n = m$ form. Divide both sides by 9.

$$1.028^x = \frac{P}{9}$$

Apply the definition of logarithm. The base is 1.028.

$$\log_{1.028}\left(\frac{P}{9}\right) = x$$

## Evaluating Logarithms to Bases Other Than 10

The methods of evaluating logs and solving equations of logs with bases other than 10 are very similar to the methods used with common logarithms. For example, when $a$ is a known power of the base $b$, $\log_b a$ can be quickly found without a calculator.

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### Example 2

**Evaluate the following.**

a. $\log_7 49$ b. $\log_8 2$ c. $\log_4\left(\frac{1}{64}\right)$

**Solution**

a. Let $\log_7 49 = x$.

$7^x = 49$ Definition of logarithm

$7^x = 7^2$ Rewrite 49 as a power of 7.

$x = \underline{\ ?\ }$ Equate the exponents.

So, $\log_7 49 = \underline{\ ?\ }$. Transitive Property of Equality

b. Let $\log_8 2 = x$.

$8^x = 2$ $\underline{\ ?\ }$

$(2^3)^x = 2^{\underline{\ ?\ }}$ Rewrite both sides as powers of 2.

$2^{\underline{\ ?\ }} = 2^{\underline{\ ?\ }}$ Power of a Power Postulate

$\underline{\ ?\ } = \underline{\ ?\ }$ Equate the exponents.

$x = \underline{\ ?\ }$ $\underline{\ ?\ }$

So, $\log_8 2 = \underline{\ ?\ }$.

c. Let $\log_4\left(\frac{1}{64}\right) = x$.

$\underline{\ ?\ } = \frac{1}{64}$

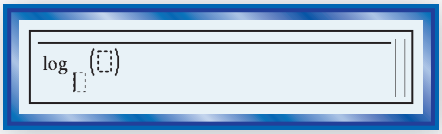
$\underline{\ ?\ } = 4^{\underline{\ ?\ }}$

$\underline{\ ?\ } = \underline{\ ?\ }$

So, $\log_4\left(\frac{1}{64}\right) = \underline{\ ?\ }$.

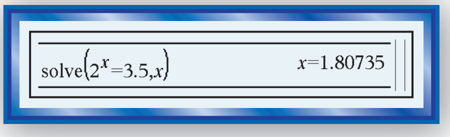
In each part of Example 2 you moved from an expression of the form $b^m = b^n$ to one of the form $m = n$. We call this "equating the exponents." When $b \neq 1$ and $b$ is positive, this is a valid process because the exponential function with base $b$ takes on a unique value for each exponent.

When $a$ is not an integer power of $b$, a CAS can be used to find $\log_b a$. Some CAS allow you to enter any base $b$ when you press the logarithm key. The template used for logs on one CAS is shown at the left below. The CAS response to $\log_2 3.5$ is shown at the right below.





In addition, any CAS may be used to find $x$ when $\log_b a = x$. Simply rewrite the equation in exponential form and solve for $x$. This method is used to find $\log_2 3.5$ as shown at the right.

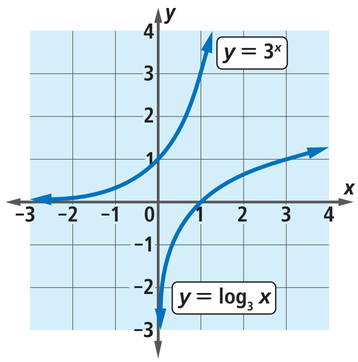



STOP **QY2**

> **QY2**
> **a.** Use a CAS to estimate $\log_7 124$ to five decimal places.
> **b.** To what exponential equation is the result of Part a an estimated solution?

## Graphs of Logarithm Functions

Both the exponential equation $3^y = x$ and the logarithmic equation $y = \log_3 x$ describe the *inverse* of the function with equation $y = 3^x$. These functions are graphed at the right. In general, the **logarithm function with base *b***, $g(x) = \log_b x$, is the inverse of the exponential function with base $b$, $f(x) = b^x$.

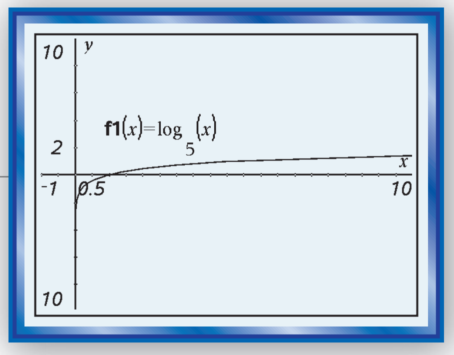



Recall that the domain of the exponential function with equation $y = 3^x$ is the set of all real numbers. Consequently, the range of the logarithm function with equation $y = \log_3 x$ is also the set of all real numbers. So logarithms to the base 3 can be negative. However, the range of the exponential function $y = 3^x$ and the domain of the corresponding logarithm function is the set of positive real numbers. This means that in the set of real numbers there is no logarithm of a nonpositive number.

## Activity

**Work with a partner and use a graphing utility.**

**Step 1** Each partner should choose a different value of $b$, with $b > 0$ and $b \neq 1$, and graph $y = \log_b x$. An example is shown at the right.




**Step 2** Fill in a chart like the one on the next page to record the domain, range, intercepts, and asymptotes of the graph for your value of $b$. Then have your partner fill in another row of the chart for his or her value of $b$.

| $b$ | Domain | Range | $x$-intercepts | $y$-intercepts | Asymptotes |
|---|---|---|---|---|---|
| ? | ? | ? | ? | ? | ? |

**Step 3** Set the window on your graphing utility to square. Each partner should graph $y = b^x$ on the same set of axes as in Step 1. What is the relationship between the graph of $y = \log_b x$ and the graph of $y = b^x$?

**Step 4** Both partners should repeat Steps 1–3 for two additional values of $b$. Choose values that are different from your partner's values, and be sure to choose some noninteger values for $b$.

**Step 5** Summarize your results. What properties do these functions seem to share?

The Activity shows that there are properties shared by all logarithm functions, regardless of their base. Every function with an equation of the form $y = \log_b x$, where $b > 0$ and $b \neq 1$, has the following properties.

1. Its domain is the set of positive real numbers.
2. Its range is the set of all real numbers.
3. Its $x$-intercept is 1, and there is no $y$-intercept.
4. The $y$-axis $(x = 0)$ is an asymptote to the graph.
5. Its graph is the reflection image over $y = x$ of the graph of $y = b^x$.

You may wonder why $b$ cannot equal 1 in the definition of the logarithm of $a$ with base $b$. It is because the inverse of the function with equation $y = 1^x$ is not a function.

STOP **QY3**

**QY3**

Why can the base $b$ of logarithms never be negative?

## Solving Logarithmic Equations

To solve a logarithmic equation, it often helps to use the definition of logarithm to rewrite the equation in exponential form.

### Example 3

Solve for $h$: $\log_4 h = \frac{3}{2}$.

**Solution**

$4^{\frac{3}{2}} = h$ definition of logarithm

$8 = h$ Rational Exponent Theorem

To solve for the base in a logarithmic equation, apply the techniques you learned in Chapters 7 and 8 for solving equations with $n$th powers.

### Example 4

Find $w$ if $\log_w 1024 = 10$.

**Solution**

| | |
|---|---|
| $w^{10} = 1024$ | definition of logarithm |
| $(w^{10})^{\frac{1}{10}} = (1024)^{\frac{1}{10}}$ | Raise both sides to the $\frac{1}{10}$ power. |
| $w = 2$ | Power of a Power Postulate |

**Check** Does $\log_2 1024 = 10$? Does $2^{10} = 1024$? Yes, it checks.

## Questions

### COVERING THE IDEAS

1. Logarithms to the base $b$ arose from Euler's attempt to describe the solution(s) to what equation?
2. **Fill in the Blank** Suppose $b > 0$ and $b \neq 1$. When $b^n = m$, $n = \underline{\ ?\ }$.
3. Write the equivalent logarithmic form for $\left(6\sqrt{3}\right)^8 = 136{,}048{,}896$.

**In 4 and 5, write the equivalent exponential form for the sentence.**

4. $\log_4 0.0625 = -2$
5. $\log_b a = c$
6. Write the inverse of the function with equation $y = 2^x$ as
   a. an exponential equation.
   b. a logarithmic equation.
7. State the domain and range of the function defined by $y = \log_3 x$.
8. Write an equation for the asymptote to the graph of $y = \log_b x$, where $b > 0$ and $b \neq 1$.
9. Sketch the graph of $y = \log_4 x$.

**In 10–12, write the corresponding exponential form of each logarithmic equation, then solve for the exponent on a CAS.**

10. $\log_{121} 1331 = x$
11. $\log_{\frac{1}{4}}\left(\frac{1}{64}\right) = y$
12. $\log_{\sqrt{5}} 625 = z$

**In 13–17, write the corresponding exponential form of each logarithmic equation. Then, solve for the given variable.**

13. $\log_a 8 = \frac{1}{3}$
14. $\log_7 c = 4$
15. $\log_{47} d = -0.2$
16. $\log_t\left(\frac{1}{6}\right) = -\frac{1}{3}$
17. $\log_w w = 1$

## APPLYING THE MATHEMATICS

18. a. Make a table of values for $y = 2^x$ with $x = -2, -1, 0, 1, 2$ and a corresponding table of values for $y = \log_2 x$.
 b. Graph $y = 2^x$ and $y = \log_2 x$ on the same set of axes.
 c. **True or False** The domain of $y = 2^x$ is the range of $y = \log_2 x$.

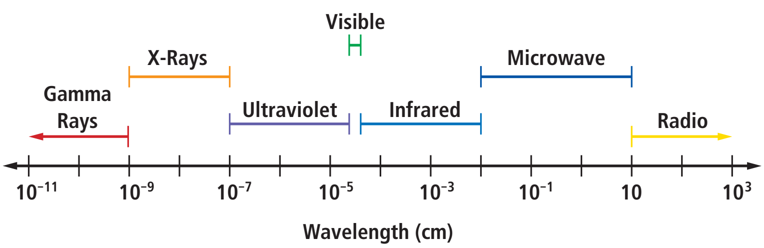
19. *Self-information* $I$ is a measure (in bits) of how the knowledge gained about a certain event adds to your overall knowledge. A formula for self-information is $I = \log_2\left(\frac{1}{x}\right)$, where $x$ is the probability of a certain event occurring.
 a. When flipping a coin, the probability of it landing on tails is 0.5. Find the number of bits this adds to your self-information.
 b. The probability of drawing a card with a diamond on it from a standard deck of cards is 0.25. Find the number of bits this adds to your self-information.
 c. If the self-information added is 2.3 bits, what was the probability of the event?

20. The depreciation of a certain automobile that initially costs \$25,000 is given by the formula $N = 25{,}000(0.85)^t$, where $N$ is the current value after $t$ years.
 a. Write this equation in logarithmic form.
 b. How old is a car that has a current value of \$9500?

21. a. Evaluate $\log_6 216$ and $\log_{216} 6$.
 b. Evaluate $\log_7 49$ and $\log_{49} 7$.
 c. Generalize the results of Parts a and b.

## REVIEW

**In 22 and 23, refer to the representation of the electromagnetic spectrum below.** (Lesson 9-6)




22. What kind of scale is used to depict the wavelengths for the electromagnetic spectrum?

23. The shortest radio wave is how many times the length of the longest gamma ray wave?

**In 24 and 25, consider the Palermo Technical Impact Hazard Scale, a logarithmic scale used by astronomers to rate the potential hazard of impact of near-Earth objects (NEO) such as asteroids, comets, and large meteoroids. A Palermo value $P$ is given by the formula $P = \log\left(\frac{p_i}{f_B T}\right)$, where $p_i$ = the event's probability, $T$ = the time in years until the event, and $f_B = 0.03E^{-0.8}$ is the annual probability of an impact event with energy $E$ (in megatons of TNT) at least as large as the event in question.** (Lesson 9-6)

While most meteor showers are caused by comet debris, the Geminid meteor showers are caused by asteroid 3200 Phaethon.

**24.** NASA keeps a database of all near-Earth objects. Discovered in 2008, the asteroid 2008 HJ is expected to approach Earth in 2077. For that year, it has an impact probability of $1.3 \cdot 10^{-6}$ and an expected impact energy of 0.408 megaton. Calculate the 2020 Palermo value for 2008 HJ.

**25.** 2002 RB182 is an asteroid with a 2005 Palermo value of –7.17. If it were to collide with Earth in 2044, it would have an expected energy of 35.6 megatons. What is the probability that it will impact Earth in 2044?

**In 26 and 27, evaluate using powers of 10.** (Lesson 9-5)

**26.** $\log 10^7$

**27.** $\log 0.0001$

**28.** Thorium-232 ($^{232}$Th) has a half-life of $1.41 \times 10^{10}$ years. (Lesson 9-2)

**a.** Write an equation giving the percent $p$ of thorium-232 in a substance after $n$ years.

**b.** Sketch a graph of your equation from Part a.

**c.** Calculate the number of years needed for 99% of the original thorium to be present.

**29.** Simplify each expression. (Lessons 8-4, 7-2)

**a.** $x^3 \cdot x^9$ **b.** $\frac{x^3}{x^9}$ **c.** $(x^3)^9$ **d.** $\sqrt[3]{x^9}$

## EXPLORATION

**30.** Logarithms were invented by English mathematician John Napier (1550–1617), in the early 1600s. Henry Briggs, also from England, first used common logarithms around 1620. In England even today logs to the base 10 are sometimes called Briggsian logarithms. Find more information about one of these mathematicians.

### QY ANSWERS

**1.** $\log_7 2401 = 4$

**2.** a. 2.47713
b. $7^x = 124$

**3.** Exponential functions are not defined for negative bases.