Lesson 9-6 Logarithmic Scales

Vocabulary

logarithmic scale

decibel, dB

BIG IDEA A **logarithmic scale** is one in which the numbers are written as the powers of a fixed number, and the exponents of the powers are used as the scale values.

Logarithmic scales are used when all the measures of an attribute are positive and cover a wide range of values from small to very large. Two logarithmic scales you may have heard of are the *decibel scale* that measures relative sound intensity and the *Richter magnitude scale* that measures the intensity of earthquakes.

Mental Math

Simplify.

a. $\frac{10^6}{10^2}$

b. $\frac{4.3 \cdot 10^4}{4.3 \cdot 10^{-7}}$

c. $\frac{2.42 \cdot 10^{12}}{1.21 \cdot 10^3}$

d. $\frac{a \cdot 10^m}{b \cdot 10^n}$

The Decibel Scale

The decibel scale is based on *watts.* The watt is a measure of power. The quietest sound that a human ear can pick up has an intensity of about 10^{-12} watts per square meter $\left(\frac{\text{w}}{\text{m}^2}\right)$. The human ear can also hear sounds with an intensity as large as $10^2 \frac{\text{w}}{\text{m}^2}$. Because the range from 10^{-12} to 10^2 is so large, it is convenient to use a measure that is based on the exponents of 10 in the intensity. This measure of relative sound intensity is the **decibel**, abbreviated **dB**. Because it is based on a ratio of units, the decibel is a dimensionless unit, like angle measure. A formula that relates the sound intensity N in $\frac{\text{w}}{\text{m}^2}$ to its relative intensity D in decibels is

$$D = 10 \log\left(\frac{N}{10^{-12}}\right).$$

The chart at the right gives the sound intensity in $\frac{\text{w}}{\text{m}^2}$ and the corresponding decibel values for some common sounds.

Sound Intensity (watts/square meter)		Relative Intensity (decibels)
10^2		140
10^1	jack hammer	130
10^0	rock concert	120
10^{-1}	ambulance siren	110
10^{-2}	school dance	100
10^{-3}	electric drill	90
10^{-4}	lawn mower	80
10^{-5}	hair dryer	70
10^{-6}	normal conversation	60
10^{-7}	refrigerator	50
10^{-8}	library	40
10^{-9}		30
10^{-10}	whisper at 5 ft	20
10^{-11}	normal breathing	10
10^{-12}	barely audible	0

Notice that as the decibel values in the right column increase by 10, corresponding intensities in the left column are multiplied by 10. Thus, if the number of decibels increases by 20, the sound intensity multiplies by $10 \cdot 10$ or 10^2. If you increase the number of decibels by 40, you multiply the watts per square meter by $10 \cdot 10 \cdot 10 \cdot 10$, or 10^4.

In general, an increase of n dB multiplies the sound intensity by $10^{\frac{n}{10}}$.

STOP QY

Example 1

Due to a law in France, manufacturers had to limit the sound intensity of their MP3 players to 100 decibels. How many times as intense is a particular MP3 player's maximum 115-decibel intensity than the limited intensity imposed in France?

Solution Use the generalization above. An increase of n dB multiplies the sound intensity by $10^{\frac{n}{10}}$. The difference in the decibel level is $115 - 100 = 15$ dB. So, the maximum sound intensity is $10^{\frac{15}{10}} = 10^{1.5} \approx 32$ times as intense as the limit imposed in France.

Experiments have shown that people perceive a sound that is 10 decibels louder than another sound as only twice as loud. That is, the ear perceives a sound of 80 decibels as only sixteen times as loud as one of 40 decibels, even though in fact the sound is 10^4 times as intense. Even if a person does not feel pain, long or repeated exposure to sounds at or above 85 decibels can cause permanent hearing loss.

Using the formula on the previous page, if you know the intensity of a sound, you can find its relative intensity in decibels.

Example 2

Grunting while hitting the ball has become a controversial issue in professional tennis. Some people are concerned that such loud sounds are unfair distractions to the opposing player. Serena Williams's grunts have been measured at a sound intensity of $6.31 \cdot 10^{-4} \frac{\text{w}}{\text{m}^2}$. Find the relative intensity of the sound in decibels.

Solution Substitute $6.31 \cdot 10^{-4}$ for N in the formula $D = 10 \log \frac{N}{10^{-12}}$.

$$D = 10 \log\left(\frac{6.31 \cdot 10^{-4}}{10^{-12}}\right) = 10 \log 631{,}000{,}000 \approx 88$$

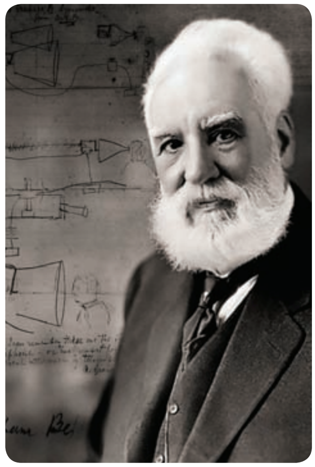
The relative intensity of the grunting is about 88 dB.

Check Refer to the chart on the previous page.
Notice that $10^{-4} < 6.31 \cdot 10^{-4} < 10^{-3}$, and the relative intensity of 88 dB falls between 80 and 90 dB. It checks.

To convert from the decibel scale to the $\frac{\text{w}}{\text{m}^2}$ scale, you can solve a logarithmic equation.

▶ QY

How many times as intense is a sound of 90 decibels as a sound of 60 decibels?



A decibel is $\frac{1}{10}$ of a bel, a unit named after Alexander Graham Bell (1847–1922), the inventor of the telephone.

Example 3

The maximum sound intensity of Melody's MP3 player is 115 dB. What is the maximum sound intensity in watts per square meter?

Solution 1

$115 = 10 \log\left(\frac{N}{10^{-12}}\right)$ Substitute 115 for D in the formula.

$11.5 = \log\left(\frac{N}{10^{-12}}\right)$ Divide each side by 10.

$10^{11.5} = \frac{N}{10^{-12}}$ Definition of common logarithm

$10^{11.5} \cdot 10^{-12} = N$ Multiply each side by 10^{-12}.

$10^{-0.5} = N$ Product of Powers Postulate

$10^{-0.5} \approx 0.316$. So, the maximum sound intensity of the MP3 player is about $0.316 \ \frac{\text{w}}{\text{m}^2}$.

Check Solve on a CAS. It checks as shown on the calculator display at the right.

$\text{solve}\left(115=10\cdot\log_{10}\left(\frac{n}{10^{-12}}\right),n\right)$

$n=.316228$

Earthquake Magnitude Scales

The most popular scale used by the media for describing the magnitudes of earthquakes is the Richter scale, designed by Charles Richter in 1935. Like the decibel scale, the Richter scale is a logarithmic scale. A value x on the Richter scale corresponds to a measured force, or amplitude, of $k \cdot 10^x$, where the constant k depends on the units being used to measure the quake. Consequently, each increase of 1 on the Richter scale corresponds to a factor of 10 change in the amplitude. The table below gives a brief description of the effects of earthquakes of different magnitudes.

Richter Magnitute	Possible Effects
1–2	Usually not felt except by instruments
3	May be felt but rarely causes damage
4	Like vibrations from heavy traffic
5	Strong enough to wake sleepers
6	Walls crack, chimneys fall
7	Ground cracks, houses collapse
8	Few buildings survive, landslides

When $k = 1$, the Richter magnitude is the common logarithm of the force, because x is the common logarithm of 10^x.

GUIDED

Example 4

Just before a 1989 World Series game in San Francisco, California there was an earthquake in the nearby Santa Cruz Mountains that registered 6.9 on the Richter scale. The tsunami that hit twelve Indian Ocean nations in December 2004 was triggered by an earthquake that measured 9.3 on the Richter scale. The force of the 2004 earthquake was how many times the force of the 1989 quake?

The 1989 earthquake in San Francisco caused the third floor of this apartment building to collapse onto this car.

Solution The amplitude of the 1989 World Series earthquake was $k \cdot 10^{\underline{\ ?\ }}$. The amplitude of the 2004 quake was $k \cdot 10^{\underline{\ ?\ }}$.

Divide these quantities to compare the amplitudes. Since we are assuming the forces were measured in the same units, the two *k*-values are the same.

$\dfrac{k \cdot 10^{\underline{\ ?\ }}}{k \cdot 10^{\underline{\ ?\ }}} = 10^{\underline{\ ?\ }}$.

As a decimal, $10^{\underline{\ ?\ }} \approx \underline{\ ?\ }$, indicating that the 2004 earthquake had about $\underline{\ ?\ }$ times as much force as the $\underline{\ ?\ }$ earthquake.

Some other examples of logarithmic scales include the pH scale for measuring the acidity or alkalinity of a substance, the scales used on radio dials, and the scale for measuring the magnitude (brightness) of stars.

Questions

COVERING THE IDEAS

In 1 and 2, use the formula $D = 10 \log\left(\frac{N}{10^{-12}}\right)$ that relates relative intensity D in dB of sound to sound intensity N in $\frac{\text{w}}{\text{m}^2}$.

1. Find the relative intensity in decibels of an explosion that has a sound intensity of $2.45 \cdot 10^{-5}$.
2. Find the intensity in $\frac{\text{w}}{\text{m}^2}$ of a sound that has a relative intensity of 78 decibels.

In 3–5, refer to the chart of sound intensity levels from the beginning of this lesson.

Serena Williams

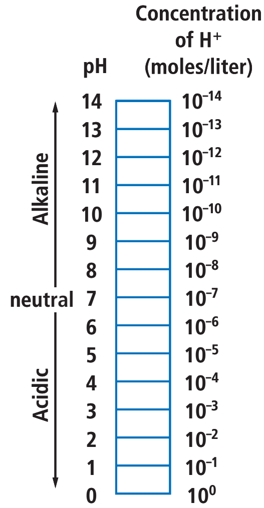
3. In Example 2, we found that Serena Williams's grunts reach an intensity of 88 decibels. Between which two powers of 10 is the equivalent sound intensity?
4. If you are near a refrigerator, how many times as intense would the sound have to be in order for it to reach the level emitted by a jackhammer?
5. How many times as intense is a whisper than a noise that is barely audible?
6. Lester Noyes is deciding between two dishwashers for purchase. One dishwasher's rating is 56 decibels (dB) and the other is 59 dB. The salesman says the sound intensity of the 59 dB dishwasher is only 5% more than the other one, and this difference is insignificant. Do you agree? Explain.
7. The grunts of the tennis player Maria Sharapova have been measured at 100 dB. How many times as loud as 88 dB is 100 dB?
8. An earthquake measuring 6.5 on the Richter scale carries how many times as much force as one measuring 5.5?
9. An earthquake measuring 8.2 on the Richter scale carries how many times as much force as one measuring 6.7?
10. If one earthquake's Richter value is 0.4 higher than another, how many times as much force is in the more powerful earthquake?

APPLYING THE MATHEMATICS

In 11 and 12, use this information. In 1979, seismologists introduced the *moment magnitude scale* based on the formula $M_w = \frac{\log M_0}{1.5} - 10.7$ for moment magnitude M_w. Seismic moment M_0 is a measure of the size of the earthquake. Its units are dyne-cm. (One dyne is the force required to accelerate a mass of 1 gram at a rate of 1 centimeter per second squared.)

11. An earthquake had a seismic moment of 3.86×10^{27} dyne-cm. Find its moment magnitude M_w.
12. If the moment magnitude of one earthquake is 1 higher than that of another, how do their seismic moments differ?

In 13–16, use the pH scale for measuring acidity or alkalinity of a substance shown at the right. Its formula is $pH = -\log(H^+)$, where H^+ is the concentration of hydrogen ions in $\frac{\text{moles}}{\text{liter}}$ of the substance.



13. The pH of normal rain is 5.6 and the pH of acidic rain is 4.3. The concentration of hydrogen ions in acid rain is how many times the concentration of hydrogen ions in normal rain?

14. For a healthy individual, human saliva has an average pH of 6.4.

a. Is saliva acidic or alkaline?

b. What is the concentration of hydrogen ions in human saliva?

15. The concentration of hydrogen ions in a typical piece of white bread is $3.16 \times 10^{-6} \frac{\text{mol}}{\text{liter}}$. What is the pH of white bread?

16. The Felix Clean Company manufactures soap. In one advertisement the company claimed that its soap has a tenth of the pH of the competing brand.

a. What range of values could the pH of the soap have if this were true?

b. What would happen if the soap did indeed have a pH as claimed in the advertisement?

c. What did the advertiser probably mean?

In 17 and 18, use this information: In astronomy, the *magnitude* (brightness) m of a star is measured not by the energy I meeting the eye, but by its logarithm. In this scale, if one star has radiation energy I_1 and absolute magnitude m_1, and another star has energy I_2 and absolute magnitude m_2, then $m_1 - m_2 = -2.5 \log\left(\frac{I_1}{I_2}\right)$.

17. The star Rigel in the constellation Orion radiates about 150,000 times as much energy as the Sun. The Sun has absolute magnitude 4.8. If the ratio of intensities is 150,000, find the absolute magnitude of Rigel.

18. Suppose the difference $m_1 - m_2$ in absolute magnitudes of two stars is 5. Find $\frac{I_1}{I_2}$, the ratio of the energies they radiate.

REVIEW

In 19 and 20, explain how to evaluate without using a calculator. (Lesson 9-5)

19. $\log_{10} 1{,}000{,}000$

20. $\log_{10} 10^{-5}$

In 21 and 22, solve. (Lesson 9-5)

21. $\log x = 7$

22. $\log 7 = x$

23. Give two equations for the inverse of the function with the equation $y = 10^x$. **(Lesson 9-5)**

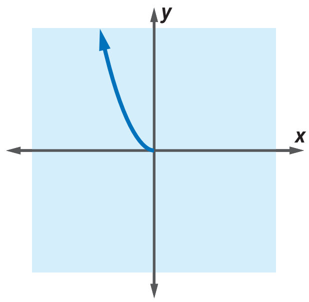
24. The *Haugh unit* is a measure of egg quality that was introduced in 1937 in the *U.S. Egg and Poultry Magazine*. The number U of Haugh units of an egg is given by the formula

$$U = 100\log\left[H - \frac{1}{100}\sqrt{32.2}\,(30W^{0.37} - 100) + 1.9\right],$$

where W is the weight of the egg in grams, and H is the height of the albumen in millimeters when the egg is broken on a flat surface. Find the number of Haugh units of an egg that weighs 58.8 g and for which $H = 6.3$ mm. **(Lesson 9-5)**

25. Is $y = 2^{-x}$ the equation for a function of exponential growth or exponential decay? Explain how you can tell. **(Lesson 9-2)**

26. Consider the graph of the function g at the right. Give the domain and range of g^{-1}. **(Lesson 8-2)**



In 27 and 28, assume all variables represent positive real numbers. Simplify. (Lessons 7-7, 7-2)

27. $\dfrac{p^5q^4}{(pm)^3}$

28. $(r^2)^{\frac{1}{4}}(t^{10})^{\frac{3}{5}}$

EXPLORATION

29. The mathematician L.F. Richardson classified conflicts according to their magnitude, the base-10 logarithm of the total number of deaths. For example, a war in which there were 10,000 deaths would have a magnitude of 4 because $10^4 = 10{,}000$. A gang fight with 10 casualties would have a magnitude of 1 because $10^1 = 1$. Use Richardson's scale to classify the Revolutionary War, the Civil War, World War I, World War II, the Vietnam War, and the Persian Gulf War. Comment on the effectiveness of Richardson's scale in comparing the number of deaths in the wars.

QY ANSWER

$10^3 = 1000$ times as intense