Lesson 9-5 Common Logarithms

BIG IDEA When a number is written as a power of 10, the exponent in the expression is the logarithm of the number to the base 10.

Vocabulary

logarithm of x to the base 10, log of x to the base 10, log base 10 of x

common logarithm, common log

logarithm function to the base 10, common logarithm function

Mental Math

Find the exact geometric mean of the arithmetic mean, median, and mode of the data set {1, 1, 2, 3, 5}.

Whenever there is a situation described by an equation of the form $y = b^x$, you may know the values of y and b, and want to find x. For instance, people may want to know when a population will reach a certain level, or people may want to know the age of an ancient Egyptian mummy. Recall that an equation for the inverse of a function can be found by switching x and y. So, $x = b^y$ is an equation for the inverse of the exponential function. You can use this equation and what you know about exponential functions to solve exponential equations.

Logarithms to the Base 10

Consider the equation $x = b^y$ when $b = 10$. Then $x = 10^y$, and we say that y is the *logarithm of x to the base 10*.

Definition of Logarithm of x to the Base 10

y is the **logarithm of x to the base 10**, or the **log of x to the base 10**, or the **log base 10 of x**, written $y = \log_{10} x$, if and only if $10^y = x$.

For example, $2 = \log_{10} 100$, because $10^2 = 100$. In the table below are some other powers of 10 and the related logs to the base 10.

Exponential Form	Logarithmic Form
$10^7 = 10{,}000{,}000$	$\log_{10} 10{,}000{,}000 = 7$
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$
$10^{\frac{1}{2}} = \sqrt{10}$	$\log_{10} \sqrt{10} = \frac{1}{2}$
$10^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{10}}$	$\log_{10} \frac{1}{\sqrt[4]{10}} = -\frac{1}{4}$
$10^a = b$	$\log_{10} b = a$

READING MATH

The word *logarithm* is derived from the Greek words *logos*, meaning "reckoning," and *arithmos*, meaning "number," which is also the root of the word *arithmetic*. A *logarithm* is literally a "reckoning number." Logarithms were used for hundreds of years for calculating (or "reckoning") before electronic calculators and computers were invented.

Logarithms are exponents. The logarithm of x is the exponent to which the base is raised to get x. Logarithms to the base 10 are called **common logarithms**, or **common logs**. We often write common logs without indicating the base. That is, log x means $\log_{10} x$.

Evaluating Common Logarithms

Common logarithms of powers of 10 can be found quickly without a calculator.

GUIDED

Example 1

Evaluate.

a. log 100 b. log 0.00001 c. log 1

Solution First write each number as a power of ten. Then apply the definition of a common logarithm. Remember that the logarithm is the exponent.

a. Because $100 = 10^{\underline{\ ?\ }}$, $\log_{10} 100 = \underline{\ ?\ }$.

b. You need to find n such that $10^n = 0.00001$.
Because $0.00001 = 10^{\underline{\ ?\ }}$, $\log 0.00001 = \underline{\ ?\ }$.

c. $10^0 = 1$, so log $\underline{\ ?\ } = \underline{\ ?\ }$.

STOP QY1

You can use the logarithm function in your calculator to evaluate $\log_{10} x$ for any positive number x.

QY1

a. What is $\log \sqrt{10}$?

b. What is the common logarithm of 0.001?

Example 2

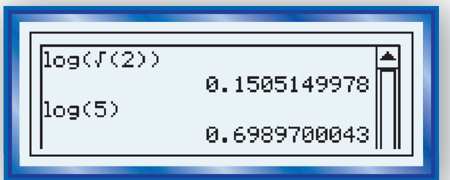
Estimate to the nearest hundred-thousandth.

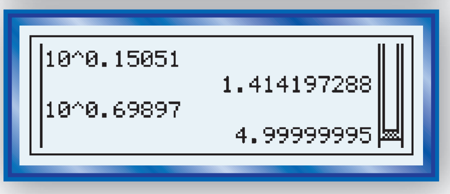
a. $\log \sqrt{2}$ b. log 5

Solution Use your calculator. You will get a display like that at the right. Round to five decimal places.

a. $\log \sqrt{2} \approx 0.15051$

b. $\log 5 \approx 0.69897$





Check Use the definition of logarithm to the base 10: $\log x = y$ if and only if $10^y = x$.

a. Does $10^{0.15051} \approx \sqrt{2}$? Use the power key on your calculator. It checks.

b. Does $10^{0.69897} \approx 5$? Yes, it checks.

STOP QY2

In general, the common logarithm of 10^x is x. That is, $\log_{10}(10^x) = x$. This is why we say a logarithm is an exponent.

The larger a number, the larger its common logarithm. Because of this, you can estimate the common logarithm of a number without a calculator. Either compare the number to integer powers of 10 or write the number in scientific notation to determine between which two consecutive integers the logarithm falls.

> **QY2**
>
> Check your answers to Example 1 using your calculator.

Example 3

Between which two consecutive integers is log 5673?

Solution 1 5673 is between $1000 = 10^3$ and $10{,}000 = 10^4$, so log 5673 is between 3 and 4.

Solution 2 $5673 = 5.673 \cdot 10^3$. This indicates that log 5673 is between 3 and 4.

Check A calculator gives $\log 5673 \approx 3.753...$. It checks.

Solving Logarithmic Equations

You can solve logarithmic equations using the definition of common logarithms.

Example 4

Solve for x.

a. $\log x = \frac{1}{2}$

b. $\log x = 0.71$

Solution

a. $\log x = \frac{1}{2}$ if and only if $10^{\frac{1}{2}} = x$.
So, $x = 10^{\frac{1}{2}} = \sqrt{10} \approx 3.162$.

b. $\log x = 0.71$ if and only if $10^{0.71} = x$. So, $x = 10^{0.71} \approx 5.129$.

The Inverse of $y = 10^x$

The inverse of the exponential function f defined by $f(x) = 10^x$ is related to common logarithms. You will find the inverse in the Activity on the next page.

Activity

MATERIALS graph paper

Step 1 Fill in the *y*-values in the table when $y = 10^x$.

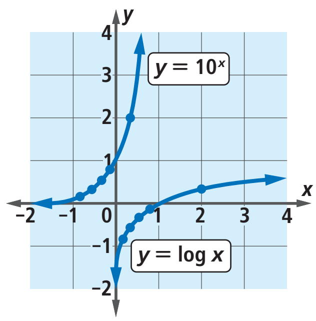
x	–2	–1	–0.75	–0.5	–0.25	0	0.25	0.5	0.75	1	2
y	?	?	?	?	?	?	?	?	?	?	?

Step 2 Plot the points $(x, 10^x)$ from the table on graph paper. Connect the points with a smooth curve.

Step 3 Plot the points of the inverse of the relation in the table on the same graph as in Step 2. Connect the points with a smooth curve.

Step 4 Graph both $y = 10^x$ and $y = \log x$ on the same set of axes in the window $-2 \le x \le 2$ and $-2 \le y \le 2$ on a graphing utility. Compare them to the graphs you made in Steps 2 and 3.

Your graphs from the Activity should look similar to the one at the right. The graph of $y = 10^x$ passes the horizontal-line test, and so its inverse is a function. This means that $y = \log x$ is an equation for a function, the inverse of the exponential function with equation $y = 10^x$.



Properties of $y = 10^x$ and $y = \log x$

The inverse of the function *f* with equation $f(x) = 10^x$ can be described in several ways as shown in the table below.

Ways of Thinking of the Inverse of *f* with $f(x) = y = 10^x$	Written	Spoken
Switching *x* and *y*	$x = 10^y$	"*x* equals 10 to the *y*th power."
Using the language of logs	$y = \log_{10} x$ $y = \log x$	"*y* equals the log of *x* to the base 10."
Using function notation	$f^{-1}(x) = \log_{10} x$ $f^{-1}(x) = \log x$	"*f* inverse of *x* equals the log of *x* to the base 10."

The curve defined by these equations is called a *logarithmic curve*. As the graph from the Activity above shows, a logarithmic curve is the reflection image of an exponential curve over the line with equation $y = x$.

The function that maps *x* onto $\log_{10} x$ for all positive numbers *x* is called the **logarithm function to the base 10**, or the **common logarithm function**.

Because they are inverses, each property of the exponential function defined by $y = 10^x$ corresponds to a property of its inverse, the common logarithm function defined by $y = \log x$.

Function	Domain	Range	Asymptote	Intercepts
$y = 10^x$	set of real numbers	set of positive real numbers	x-axis ($y = 0$)	y-intercept $= 1$
$y = \log x$	set of positive real numbers	set of real numbers	y-axis ($x = 0$)	x-intercept $= 1$

Questions

COVERING THE IDEAS

1. If $m = \log_{10} n$, what other relationship exists between m, 10, and n?

2. a. Write in words how to read the expression $\log_{10} 8$.
   b. Evaluate $\log_{10} 8$ to the nearest ten-thousandth.

In 3–8, evaluate using the definition of common logarithms.

3. $\log 10{,}000{,}000$
4. $\log 10^5$
5. $\log 0.0000001$
6. $\log \sqrt[3]{10}$
7. $\log \frac{1}{10}$
8. $\log 10$

9. Between which two consecutive integers is the value of the common logarithm of 100,000,421?

In 10–12, approximate to the nearest thousandth.

10. $\log 3$
11. $\log 0.00309$
12. $\log 309{,}000$

In 13 and 14, solve for *x*.

13. $\log x = 5$
14. $\log x = 1.25$

15. Consider the graph of $y = \log_{10} x$.
    a. Name its x- and y-intercepts, if they exist.
    b. Name three points on the graph.
    c. Name the three corresponding points on the graph of $y = 10^x$.

16. What are the domain and range of the common logarithm function?

17. **Fill in the Blank** The functions f and g, with equations $f(x) = 10^x$ and $g(x) =$ __?__, are inverses of each other.

APPLYING THE MATHEMATICS

18. If $5 \log v = 2$, what is the value of v?

19. If a number is between 10 and 100, its common logarithm is between which two consecutive integers?

20. The common logarithm of a number is –5. What is the number?

21. Evaluate $10^{\log 3.765}$.

22. Explain why for all positive numbers a, $10^{\log a} = a$.

23. If $f(x) = 10^x$ and $g(x) = \log_{10} x$, what is $f \circ g(x)$? Explain your answer.

In 24 and 25, use this information: Most of today's languages are thought to be descended from a few common ancestral languages. The longer the time lapse since a language split from the ancestral language, the fewer common words exist in the descendant language. Let c = the number of centuries since two languages split from an ancestral language. Let w = the fraction of words from the ancestral language that are common to the two descendent languages. In linguistics, the equation $\frac{10}{c} = \frac{2 \log r}{\log w}$ (in which $r = 0.86$ is the index of retention) has been used to relate c and w.

English and Spanish descended from the Indo-European language family. Chinese descended from the Sino-Tibetan family.

24. If about 15% of the words in an ancestral language are common to two different languages, about how many centuries ago did they split from the ancestral language?

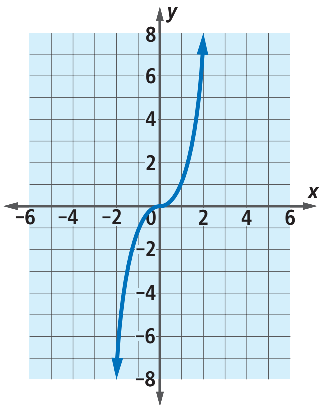
25. If it is known that two languages split from an ancestral language about 1500 years ago, about what percentage of the words in the ancestral language are common to the two languages?

REVIEW

26. Find an equation of the form $y = ab^x$ that passes through the points (0, 3) and (4, 48). **(Lesson 9-4)**

27. A tool and die machine used in a metal working factory depreciates so that its value after t years is given by $N(t) = Ce^{-0.143t}$, where C is its initial value. If after 4 years the machine is worth \$921,650, what was its original value? **(Lesson 9-3)**

28. Refer to the graph at the right of the cubing function f defined by $f(x) = x^3$. **(Lessons 8-3, 7-6, 7-1)**



a. What are the domain and range of this function?

b. Graph the inverse.

c. Is the inverse a function? Why or why not?

d. What name is usually given to the inverse function?

In 29 and 30, simplify without using a calculator. (Lessons 7-6, 7-3, 7-2)

29. $(13^3 \cdot 13^{-6})^2$

30. $\sqrt[5]{8^{15}}$

EXPLORATION

31. In 2005, the three leading international public repositories for DNA and RNA sequence information reached over 100 gigabases (100,000,000,000 bases) of sequence. For perspective, the human genome is about 3 gigabases. The table below presents the approximate size of the international databases in each year from 2000-2005.

Year	2000	2001	2002	2003	2004	2005
Gigabases	11	18	37	53	81	104

a. Write an equation to predict the number n of bases stored t years after 2000.

b. What is the growth rate of sequence mapping?

c. Use your equation in Part a to predict the year that the international databases will hold the amount of information equal to the information contained in the genomes of 1000 people.

QY ANSWERS

1. a. $\frac{1}{2}$ b. –3

2.