Lesson 9-3 Continuous Compounding

Vocabulary

e

compounded continuously

BIG IDEA The more times that a given interest rate is compounded in a year, the larger the amount an account will earn. But there is a limit to the amount earned, and the limit is said to be the result of *continuous compounding*.

Recall the General Compound Interest Formula,

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

which gives the amount A that an investment is worth when principal P is invested in an account paying an annual interest rate r and the interest is compounded n times per year for t years.

Suppose you put \$1 into a bank account. If the bank were to pay you 100% interest compounded annually, your money would double in one year because the interest would equal the amount in the account. Using the formula with $P = 1$ dollar, $r = 100\% = 1$, $n = 1$, and $t = 1$ year, we have

$$A = 1\left(1 + \frac{1}{1}\right)^1 = 2 \text{ dollars.}$$

Now suppose you put \$1 into a bank account and the bank paid 100% compounded *semiannually*. This means that the bank pays 50% interest twice a year. Now you receive \$0.50 in interest after six months, giving a total of \$1.50. Then, after another six months you receive \$0.75 in interest on the \$1.50, giving a total after one year of \$2.25. This agrees with what you would compute using the General Compound Interest Formula with $n = 2$. Then,

$$A = \left(1 + \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25 \text{ dollars.}$$

You are asked to explore what happens as n gets larger in the following Activity.

Mental Math

Consider the largest sphere that will fit in a cube with side length 6.

a. Find the exact volume of the sphere.

b. How many spheres of diameter 3 will fit in the cube without overlapping?

Activity

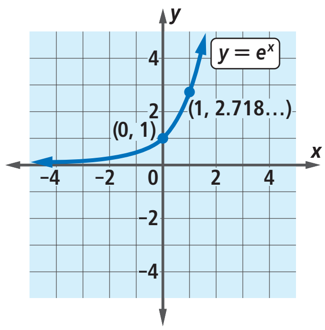
Complete a table like the one on the next page to show the value of \$1 at the end of one year ($t = 1$) after an increasing number n of compounding periods per year at a 100% annual rate.

(continued on next page)

Compounding Frequency	n = Number of Compoundings per Year	$P\left(1+\frac{r}{n}\right)^{nt}$	A
Annually	1	$1\left(1+\frac{1}{1}\right)^1$	\$2.00
Semiannually	2	$1\left(1+\frac{1}{2}\right)^2$	\$2.25
Quarterly	4	?	\$2.44141
Monthly	?	?	?
Daily	?	?	?
Hourly	?	?	?
By the Minute	?	?	?

The Activity shows that the more frequently a bank compounds, the more your earnings will be. The sequence of values for the total amount gets closer and closer to the number **e**, which is approximately equal to 2.71828. We say that e is the value of \$1 after one year invested at 100% interest **compounded continuously**.

The number e is named after Euler, who proved that the sequence of numbers of the form $\left(1+\frac{1}{n}\right)^n$ approaches this particular number as n increases. Like π, e is an irrational number that can be expressed as an infinite, nonrepeating decimal. The following are the first 50 digits of the decimal expansion for e, and a graph of $y = e^x$ is shown at the right.



$e \approx 2.71828\,18284\,59045\,23536\,02874\,7135\,2\,66249\,77572\,47093\,69995\ldots$

Like π, an approximation to e is stored on virtually every calculator.

STOP **QY1**

> **QY1**
>
> What value does your calculator display for e? (Hint: Look for a key or menu selection labeled e^x, and evaluate e^1.)

Interest Compounded Continuously

Of course, savings institutions do not pay 100% interest. But the number e appears regardless of the interest rate.

Consider an account in which 4% interest is paid on \$1 for one year. The table at the right shows some values of A for different compounding periods. As n increases, the total amount gets closer and closer to \$1.040810…, the decimal value of $e^{0.04}$. Furthermore, in t years of continuous compounding at this rate, the dollar would grow to $(e^{0.04})^t$, or $e^{0.04t}$. So, if an amount P were invested, the amount would grow to $Pe^{0.04t}$.

Compounding Frequency	$P\left(1+\frac{r}{n}\right)^{nt}$	A
Annually	$1\left(1+\frac{0.04}{1}\right)^1$	\$1.04
Quarterly	$1\left(1+\frac{0.04}{4}\right)^4$	\$1.040604
Daily	$1\left(1+\frac{0.04}{365}\right)^{365}$	\$1.040808

In general, for situations where interest is compounded continuously, the general Compound Interest Formula can be greatly simplified.

Continuously Compounded Interest Formula

If an amount P is invested in an account paying an annual interest rate r compounded continuously, the amount A in the account after t years is

$$A = Pe^{rt}.$$

STOP QY2

QY2

What will an account with an initial deposit of $1000 be worth after 3 years if interest is compounded continuously at 5% annual interest rate?

Example 1

Talia invested $3,927.54 in a zero-coupon bond paying 5.5% compounded semiannually. After 30 years the value of the bond will be $20,000. How much would Talia's investment have been worth after 30 years if interest were compounded continuously instead of twice per year?

Solution Use $A = Pe^{rt}$, with $P = 3927.54$, $r = 0.055$, and $t = 30$.

$A = 3927.54e^{0.055(30)}$

$A = 3927.54e^{1.65} \approx 20{,}450.62$

After 30 years, the bond would be worth $20,450.62. This is $450.62 more than a bond with semiannual compounding would earn.

Other Uses of the Number *e*

Many formulas for continuous growth and decay are written using e as the base because the exponential function $y = e^x$ has special properties that make it particularly suitable for applications. We do not study those properties here, but you will learn them if you study calculus. You do not need to know these properties to find values of the function.

GUIDED

Example 2

A *capacitor* is an electrical device capable of storing an electric charge and releasing that charge very quickly. For example, a camera uses a capacitor to provide the energy needed to operate an electronic flash. The percent Q of charge in the capacitor t seconds after a flash begins is given by the formula $Q = Pe^{-10.0055t}$, where P is the initial percent of charge in the capacitor. If 44.8% of the charge is left 0.08 second after a flash begins, to what percent was the capacitor originally charged?

(continued on next page)

Solution We need to find P. When $t = 0.08$, $Q = \underline{\quad ? \quad}$.
Substitute these values and solve for P.

$$Q = Pe^{\underline{\ ?\ }t}$$
$$\underline{\ ?\ } = Pe^{(\underline{\ ?\ })(\underline{\ ?\ })}$$
$$\underline{\ ?\ } \approx P(\underline{\ ?\ })$$
$$\underline{\ ?\ } \approx P$$

The capacitor was originally charged to $\underline{\ ?\ }$% of capacity.

Formulas such as $A = 3000e^{0.05t}$ and $Q = Pe^{-10.0055t}$ model exponential growth and decay, respectively. Models like these are often described using function notation. Let C be the initial amount, and let r be the growth factor by which this amount continuously grows or decays per unit time t. Then $N(t)$, the amount at time t, is given by the equation

$$N(t) = Ce^{rt}.$$

This equation can be rewritten as $N(t) = C(e^r)^t$. So it is an exponential equation of the form

$$y = ab^x,$$

where $a = C$, $x = t$, and the growth factor $b = e^r$. If r is positive, then $e^r > 1$ and there is exponential growth. If r is negative, then $0 < e^r < 1$ and there is exponential decay.

Questions

COVERING THE IDEAS

1. **Fill in the Blank** Like π, the number e is a(n) $\underline{\quad ? \quad}$ number.
2. Approximate e to the nearest ten-billionth.
3. Use the General Compound Interest Formula.
   a. What is the value of \$1 invested for one year at 100% interest compounded daily, to the nearest hundredth of a cent?
   b. As n increases, the value in Part a becomes closer and closer to what number?
4. Approximate $e^{0.055}$ to the nearest hundred-thousandth.

5. Suppose \$1000 is invested at 6% interest compounded continuously.
   a. What is its value at the end of one year?
   b. What is its value at the end of 3.5 years?

6. Suppose \$8000 is invested at an annual interest rate of 4.5% for 15 years.
   a. How much will be in the account if the interest compounds continuously?
   b. How much will be in the account if the interest compounds annually?

7. Use the formula $Q = Pe^{-10.005t}$ given in Example 2. What is the initial charge of a capacitor if 30% of its charge remains after 0.1 second?

8. Consider the function N with $N(t) = Ce^{rt}$.
   a. What does C represent?
   b. What does e^r represent?
   c. What are the domain and range of N?
   d. How can you determine whether N models exponential growth or exponential decay?

APPLYING THE MATHEMATICS

9. Equations of three functions are shown below.
   (i) $y_1 = e^x$ (ii) $y_2 = \left(\frac{1}{e}\right)^{-x}$ (iii) $y_3 = e^{-x}$
   a. Determine whether each function is increasing, decreasing, or neither on the set of real numbers.
   b. Check your answer in Part a by graphing each function for $-2 \le x \le 2$.
   c. Explain why two of the graphs in Part b coincide.
   d. Which of the functions describe(s) exponential growth?
   e. Which of the functions describe(s) exponential decay?

10. **Fill in the Blank** Write >, <, or =: π^e __?__ e^π.

11. A machine used in an industry depreciates so that its value $N(t)$ after t years is given by $N(t) = Ce^{-0.35t}$.
    a. What is the annual rate r of depreciation of the machine?
    b. If the machine is worth \$90,000 after 4 years, what was its original value?

12. The amount L of americium, a radioactive substance, remaining after t years decreases according to the formula $L = Be^{-0.0001t}$. If 5000 micrograms are left after 2500 years, how many micrograms of americium were present initially?

13. In the 1980s, researchers developed a special "inhibited growth" model to predict the growth of the U.S. population, using data from the 1960s and 1970s. The model predicts the population $p(y)$ (in millions) of the U.S. y years after 1960. An equation for the model is

$$p(y) = \frac{64{,}655.6}{179.3 + 181.3e^{-0.02676y}}.$$

a. According to this model, what is the predicted U.S. population in 2100?

b. Let $y =$ the number of years since 2000. Then a newer model for the U.S. population is $q(y) = 281.4e^{0.0112y}$. According to this model, what population is predicted for 2100?

c. Which answer is greater, the answer for Part a or Part b? Can you think of any reason to account for such differences?

d. Research last year's U.S. population and compare it to the values predicted by the two models.

REVIEW

14. Nobelium was discovered in 1958 and named after Alfred Nobel. Nobelium-255 (^{255}No) has a half-life of 3 minutes. Suppose 100% of nobelium-255 is present initially. **(Lesson 9-2)**

a. Make a table of values showing how much nobelium-255 will be left after 1, 2, 3, 4, and 5 half-life periods.

b. Write a formula for the percent A of nobelium-255 left after x half-life periods.

c. Write a formula for the percent A of nobelium-255 left after t minutes.

15. Let $f(x) = 16^x$. **(Lessons 9-1, 8-2, 7-7, 7-3)**

a. Evaluate $f(-3)$, $f(0)$, and $f\left(\frac{3}{2}\right)$.

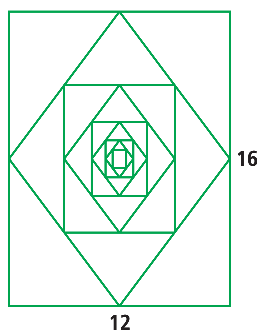
b. Identify the domain and range of f.

c. Give an equation for the reflection image of the graph of $y = f(x)$ over the line with equation $y = x$.

16. Rationalize the denominator of $\frac{2\sqrt{5}+1}{2\sqrt{5}-1}$. **(Lesson 8-6)**

17. Solve $5k^3 = 27$. **(Lessons 8-4, 7-6)**

18. Use the diagram at the right. Midpoints of a 12-by-16 rectangle have been connected to form a rhombus. Then midpoints of the rhombus are connected to form a rectangle, and so on. **(Lesson 7-5, Previous Course)**



a. List the perimeters of the first 6 rectangles.

b. What kind of sequence is formed by these perimeters?

19. For what values of a does the equation $ax^2 + 8x + 7 = 0$ have no real solutions? (**Lesson 6-10**)

20. Choose all that apply. What kind of number is $\sqrt{-25}$? (**Lessons 6-8, 6-2**)

A rational **B** irrational **C** imaginary

21. The table below gives the average height in centimeters for girls of various ages in the U.S. (**Lesson 3-5**)

Age	2	4	6	8	10
Height (cm)	85	101	115	127	138

a. Let x be age and y be height. Find an equation for the line of best fit for this data set.

b. Use the equation in Part a to predict the height of a girl at age 12.

EXPLORATION

22. Another way to get an approximate value of e is to evaluate the infinite sum

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots .$$

(Recall that $n!$ is the product of all integers from 1 to n inclusive.)

a. Use your calculator to calculate each of the following to the nearest thousandth.

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

b. How many terms must you add to approximate $e = 2.71828\ldots$ to the nearest thousandth? What is the last term you need to add to do this?

QY ANSWERS

1. Answers vary. Sample: 2.71828183

2. $1161.83