Lesson 9-2 Exponential Decay

Vocabulary

exponential decay
depreciation
half-life

BIG IDEA When the constant growth factor in a situation is between 0 and 1, exponential decay occurs.

In each exponential growth situation in Lesson 9-1, the growth factor b in $f(x) = ab^x$ was greater than 1, so the value of $f(x)$ increased as x increased. When the growth factor b is between 0 and 1, the value of $f(x)$ *decreases* as x increases. The situation then is an instance of **exponential decay**.

Mental Math

Imagine the graph of $y = 0.5x^2$. How many lines

a. are asymptotes to this graph?

b. are lines of symmetry of this graph?

c. intersect the graph in two points?

d. are neither horizontal nor vertical and intersect the graph in exactly one point?

e. are horizontal and intersect the graph in exactly one point?

Depreciation as an Example of Exponential Decay

Automobiles and other manufactured goods that are used over a number of years often decrease in value. This decrease is called **depreciation**. The function that maps the year onto the value of the car is an example of an exponential decay function.

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Example 1

Suppose a new SUV costs $36,025 and depreciates 12% each year.

a. Write an equation that gives the SUV's value when it is *t* years old.

b. Predict the SUV's value when it is 4 years old.

Solution

a. If the vehicle loses 12% of its value annually, it keeps 100% − 12% = __?__% of its value. Because each year's value is a constant multiple of the previous year's value, the value of the car after t years can be modeled by an exponential function V with equation $V(t) = ab^t$. $V(t)$ is the value of the car when it is t years old.

a is the original value, so a = __?__.

b is the constant multiplier or ratio, so b = __?__.

So, V(t) = __?__.

b. When the car is 4 years old, t = __?__.
Substitute. $V(\underline{\ ?\ }) = \underline{\ ?\ } \cdot 0.88^{\underline{\ ?\ }} \approx \underline{\ ?\ }$.
The car's value will be about __?__ after 4 years.

Half-Life and Radioactive Decay

In Lesson 9-1, you saw that when a quantity grows exponentially, its doubling time is constant. If a substance decays exponentially, the amount of time it takes for half of the atoms in this substance to decay into another matter is called its **half-life**. Half-life is an important feature of some chemical processes and of radioactive decay. The half-life of a radioactive substance can be as short as a small fraction of a second, as in the 0.002-second half-life of hassium-265, or as long as billions of years, as in the 4.47 billion-year half-life of uranium-238, a naturally occurring radioactive element.

Example 2

Carbon-14 (sometimes written as ^{14}C) has a half-life of 5730 years. This means that in any 5730-year period, half of the carbon-14 decays and becomes nitrogen-14, and half remains. Suppose that an object contains 100 g of carbon-14. Let a_n be the number of grams of carbon-14 remaining after n half-life periods.

a. How many grams of carbon-14 remain after 4 half-life periods?

b. How many years is 4 half-life periods?

c. Write a recursive formula for a_n.

d. Write an explicit formula for a_n.

e. Write a function for the amount of carbon-14 remaining after any real number of half-life periods x.

Solution

a. Make a table of values of a_n for $n = 0, 1, 2, 3$, and 4, as at the right. The initial amount of carbon-14 is $a_0 = 100$. Each successive value of a_n is one-half the previous value.

6.25 grams of carbon-14 remain after 4 half-life periods.

n	a_n
0	100
1	50
2	25
3	12.5
4	6.25

b. One half-life period = 5730 years, so,

4 half-life periods = $4 \cdot 5730 = 22{,}920$ years.

c. The amount a_n of carbon-14 remaining after n half-life periods forms a geometric sequence with a constant ratio of $\frac{1}{2}$.

A recursive formula for this sequence is

$$\begin{cases} a_0 = 100 \\ a_n = \frac{1}{2}a_{n-1}, \text{ for integers } n \geq 1. \end{cases}$$

d. Use the Explicit Formula for a Geometric Sequence from Chapter 7.

An explicit formula for this sequence is

$$a_n = 100 \cdot \left(\frac{1}{2}\right)^n \text{ for integers } n \geq 0.$$

e. Use the equation $f(x) = ab^x$ with $a = 100$ and $b = \frac{1}{2}$.

$$f(x) = 100 \cdot \left(\frac{1}{2}\right)^x$$

During a plant or animal's life, the carbon-14 in its body is naturally replenished from the environment. Once it dies, the amount of carbon-14 decays exponentially as described in Example 2. Archeologists and historians use this fact to estimate the date at which an ancient artifact made of organic material was created.

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Example 3

Suppose a sample of a piece of parchment began with 100% pure carbon-14. Use the information about carbon-14 from Example 2.

a. Write an equation for the percent of carbon-14 remaining in the sample after *x* half-life periods.

b. Graph your equation from Part a and use it to find the age of the sample to the nearest century if it contains 80% of its original carbon-14.

Parchment is an organic material, usually made of animal skin.

Solution

a. Use the exponential equation $f(x) = ab^x$. Rewrite the initial amount as a whole number: $a =$ __?__ % = __?__.

Let $f(x)$ be the amount of carbon-14 remaining in the sample after x half-life periods. Then $f(x) =$ __?__ • __?__ ^{x}.

b. Graph f on a graphing utility. Trace on the graph where $y \approx$ __?__ and record the value of x. $x \approx$ __?__. So, the piece of parchment is $\approx$ __?__ • 5730 $\approx$ __?__ years old to the nearest century if it contains 80% of its original carbon-14.

Growth versus Decay

The examples of exponential growth from Lesson 9-1 and the examples of exponential decay from this lesson fit a general model called the *Exponential Change Model*.

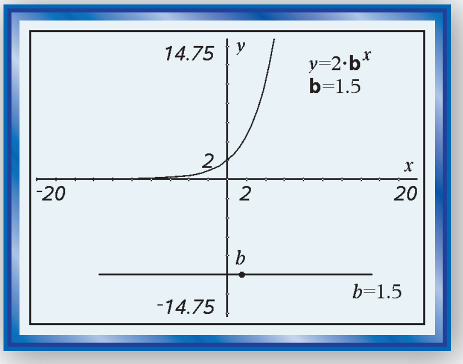
Exponential Change Model

If a positive quantity a is multiplied by b (with $b > 0$, $b \neq 1$) in each unit period, then after a period of length x, the amount of the quantity is ab^x.

Activity

MATERIALS dynamic graphing application or graphing utility

Explore how changing *a* and *b* affect the graphs of the family of functions with equations $f(x) = ab^x$. You can use a dynamic graphing application provided by your teacher or graph several instances of the family on a graphing utility.



Step 1 Graph $f(x) = ab^x$ for a constant value of $a > 0$ and different values of b between 1 and 4.

a. What features of the graph change as b changes?

b. What features of the graph stay the same? (Hint: Why are these called exponential *growth* functions?)

Step 2 Now graph $f(x) = ab^x$ for the same a as in Step 1 and different values of b between 0 and 1.

a. What features of the graph change as b changes?

b. What features of the graph stay the same? (*Hint:* Why are these called exponential *decay* functions?)

Step 3 How are the graphs, where $0 < b < 1$, different from the graphs where $b > 1$? How are the two kinds of graphs similar?

Step 4 Now graph $f(x) = ab^x$ for a constant value of $b > 0$ and some values of a between 1 and 4. How does a affect the graph of $y = ab^x$?

Step 5 Copy and complete the chart below to summarize the features of exponential growth and exponential decay functions.

Property	Exponential Decay $a > 0, 0 < b < 1$	Exponential Growth $a > 0, b > 1$
Domain	set of real numbers	set of real numbers
Range	?	?
y-intercept	?	?
x-intercept	?	?
Horizontal asymptote	?	?
As x increases, y (increases/decreases).	?	?

Step 6 Sketch a graph showing exponential growth and a graph showing exponential decay, labeling $(0, a)$ on both graphs.

Notice that exponential growth and decay graphs have many features in common. This is not surprising because every exponential growth curve is the reflection image of an exponential decay curve over the y-axis. You are asked about this property in Questions 7 and 14.

Questions

COVERING THE IDEAS

1. Refer to Example 1. If the depreciation model continues to be valid, what would the SUV be worth when it is 10 years old?

2. Suppose a car purchased for $16,000 decreases in value by 8% each year.
   a. What is the yearly growth factor?
   b. How much is the car worth after 2 years?
   c. How much is the car worth after x years?

3. Define *half-life*.

In 4 and 5, refer to Example 3.

4. What percent of an artifact's carbon-14 would remain after 8000 years?

5. In 1996, a human skeleton, nicknamed Kennewick Man, was found in the Columbia River in the state of Washington. Examination showed that the skeleton had about 32% of its original carbon-14 in its bones. In about what year did Kennewick Man die?

6. **Fill in the Blank** If $y = ab^x$, $a > 0$, and $0 < b < 1$, then y __?__ as x increases.

7. a. Sketch a graph $y = 4^x$, when $-3 \leq x \leq 3$.
   b. Sketch a graph $y = \left(\frac{1}{4}\right)^x$, when $-3 \leq x \leq 3$.
   c. Explain how the graphs in Parts a and b are related to each other.

8. **True or False** The graph of every exponential function with equation $f(x) = ab^x$ has x-intercept a. Explain your answer.

9. **True or False** The x-axis is an asymptote for the graph of an exponential function. Explain your answer.

APPLYING THE MATHEMATICS

10. According to recent estimates, in the future the population of Japan is predicted to *decrease* by 0.9% per year. In 2007, the population of Japan was about 127,433,000 people. Let $J(x)$ be the population (in millions) of Japan x years after 2007. Assume that J is an exponential function with domain $0 \le x \le 45$.

Tokyo, Japan is one of the most densely populated cities in the world.

- **a.** Find a formula for $J(x)$.
- **b.** Make a table of values of $J(x)$ every 5 years from 2007 until 2052.
- **c.** According to this model, in what year would the population of Japan first fall below 110 million people?
- **d.** Assume the population of Japan in 2008 is about 127,288,000. How well does your model predict this number? Check your model's validity with data for more recent years, if available on the Internet.

11. Suppose that a new car costs \$20,000 and one year later is worth \$18,000. Let N be the value of the car after t years.

- **a.** Assume the depreciation is exponential. Make a table of values for $t = 1, 2, 3, 4$.
- **b.** Repeat Part a if the depreciation is linear.
- **c.** Suppose you lease this car for 6 months and then decide to buy it (called *buying out the lease*). The balance to pay for the car is partly based on the car's existing value; that is, a higher value means you pay more for the car. On this basis, do you prefer that the dealer use a linear or exponential depreciation model? Explain your answer.

12. Sam Dunk has a 78% probability of making a free throw on one attempt. Assume that success or failure on one attempt does not affect the probability of success or failure on the next attempt.

- **a.** Find the probability that Sam makes 2 free throws in a row.
- **b.** Find the probability that Sam makes 3 free throws in a row.
- **c.** Let $f(n)$ be the probability that Sam makes n free throws in a row from the beginning of a game. Write a formula for $f(n)$.
- **d.** Find the smallest positive value of n such that $f(n) < 0.5$.
- **e.** What does your answer to Part d mean in the context of Sam Dunk's free throws?

13. In 1965, the computer scientist Gordon Moore first noticed that the size, speed, and cost of computing elements change exponentially over time. In 1981, a 5-megabyte hard drive cost about \$1700, or \$340 per megabyte. The cost per megabyte has decreased exponentially with a half-life of about 1.2 years since then. Let $P(x)$ be the cost of one megabyte of hard-drive storage x half-life periods after 1981.

a. Write an equation to for the exponential function P.

b. Copy and fill in the table below.

x	–2	–1	0	1	2	3
$P(x)$	?	?	?	?	?	?

c. The year 2020 corresponds to how many half-life periods since 1981? Use your model to predict the cost of hard-drive storage in 2020.

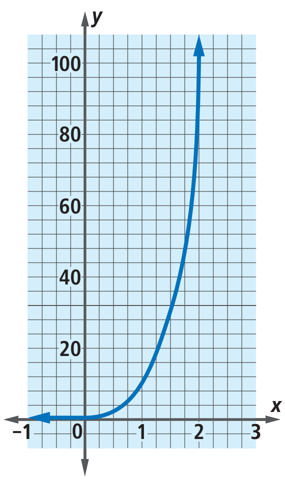
d. According to your model, when will the cost of hard-drive storage first be less than \$0.01 per *gigabyte*? (1 gigabyte = 1000 megabytes)

e. Is it realistic that one gigabyte of memory could cost less than a penny? Give a reason for your answer.

14. Suppose $g(x) = ab^x$ is an exponential growth function. Let $h(x) = g(-x)$ for all values of x. Use properties of powers to show that h is an exponential decay function. How are the graphs of h and g related?

REVIEW

15. Consider the function graphed at the right. **(Lessons 9-1, 1-2)**



a. Multiple Choice Which could be an equation for the graph?

A $f(x) = 10^{-x}$ B $g(x) = 10x$

C $h(x) = \left(\frac{1}{x}\right)^{10}$ D $j(x) = 10^x$

b. What is the domain of the function that answers Part a?

c. What is the range of the function that answers Part a?

16. In 2005 the population of Dhaka, Bangladesh, was 12,576,000, and the growth rate was 3.6% per year. If that growth rate continues, what will the population of Dhaka be in 2015? **(Lesson 9-1)**

17. Shilah is raising guppies, a popular species of freshwater aquarium fish. To raise guppies, it is important to have plenty of aquarium space, ample food, and places for babies to hide (or the guppy mothers may eat them). Shilah began with 12 guppies and the population increased at a biweekly (every two weeks) rate of 33%. **(Lessons 9-1, 7-2)**

   a. How many guppies did Shilah have after 10 weeks?

   b. What is an equation for the number g of guppies after w weeks?

18. The inflation rate, a rate at which the average price of goods increases, is reported monthly by the U.S. government. Suppose a monthly rate of 0.3% was reported for January. Assume this rate continues for a year. **(Lessons 9-1, 5-1)**

   a. What is the inflation rate for the year?

   b. The value 0.3% has been rounded. The actual value could range from 0.25% up to, but not including, 0.35%. Write an inequality for r, the annual inflation rate, based on those two extreme values.

19. Give the decimal approximation to the nearest tenth. **(Lessons 7-7, 7-2)**

   a. 5^3 b. $5^{\frac{5}{2}}$ c. $5^{\sqrt{7}}$

20. Write $\frac{10^{12.7}}{10^{9.3}}$ as a power of ten and as a decimal to the nearest thousandth. **(Lessons 7-2)**

21. Simplify $\frac{tr^{n+1}}{r}$. **(Lesson 7-2)**

EXPLORATION

22. Often we think of populations as *growing* exponentially, but as Question 10 shows, a country's population may *decrease* exponentially.

   a. Find some countries in the world whose population has been decreasing.

   b. List some factors that would explain their decreasing population.

23. In this lesson you have seen several cases in which scientists found an object with a certain percent of its carbon-14 left. Research how scientists find out how much carbon-14 was originally present.