Lesson

9-1 Exponential Growth

Vocabulary

exponential function

exponential curve

growth factor

BIG IDEA Exponential functions model situations of constant growth.

A Situation of Exponential Growth

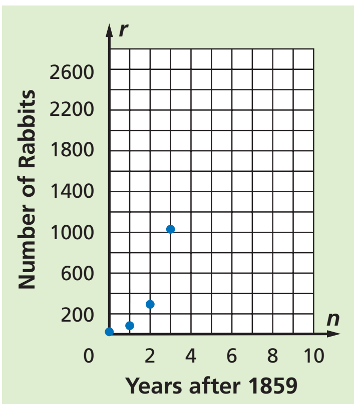
When a species is introduced to a new environment, it often has no natural predators and multiplies quickly. This situation occurred in Australia in 1859, when a landowner named Thomas Austin released 24 rabbits for hunting. The rabbits reproduced so quickly that within 20 years they were referred to as a "grey carpet" on the continent, and drove many native plant and animal species to extinction. Because a pair of rabbits can produce an average of 7 surviving baby rabbits a year when in a dense environment, you can estimate that the rabbit population multiplied by a factor of $\frac{7}{2}$, or 3.5, each year.

To model this situation, let $r_0 = 24$ be the initial population of rabbits in 1859. This is similar to using h_0 for initial height and v_0 for initial velocity in previous formulas. Then let $r_n =$ the number of rabbits n years after 1859. A recursive formula for a sequence modeling this situation is

$$\begin{cases} r_0 = 24 \\ r_n = 3.5r_{n-1}, \text{ for } n \geq 1 \end{cases}.$$

This is a geometric sequence with starting term 24 and constant ratio 3.5. The table below shows the population r_n predicted by the model for $n = 1, 2, 3, \ldots, 10$ years after 1859, rounded to the nearest whole number of rabbits. The first four ordered pairs are graphed below.

n	r_n	n	r_n
0	24	6	44,118
1	84	7	154,414
2	294	8	540,450
3	1,029	9	1,891,575
4	3,602	10	6,620,514
5	12,605		



Mental Math

Sahar is preparing for a math test. She plans to study on the four days before the test, and each day she will study $1\frac{1}{2}$ times as long as the day before. On the first day, she studies 24 minutes. How many hours and minutes is she planning to study on the fourth day?

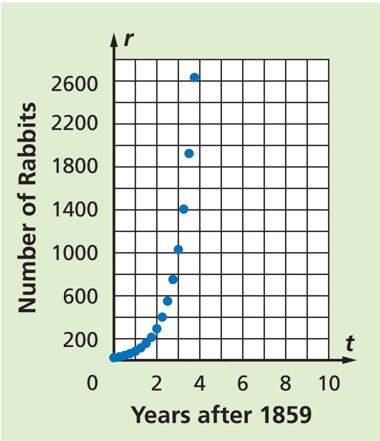
A female rabbit can give birth to several litters in one year, with up to 12 baby rabbits per litter.

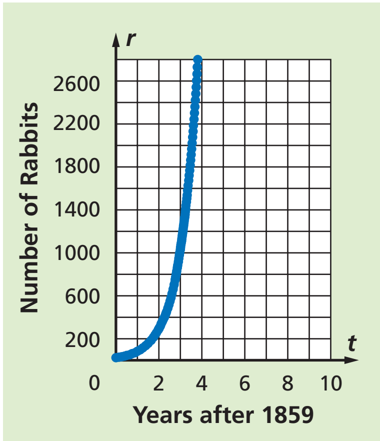
An explicit formula for this sequence is $r_n = 24(3.5)^n$ for $n \geq 0$. By representing the population as the function f with equation $r = f(t) = 24(3.5)^t$, you can estimate the population r at any real number of years $t \geq 0$.

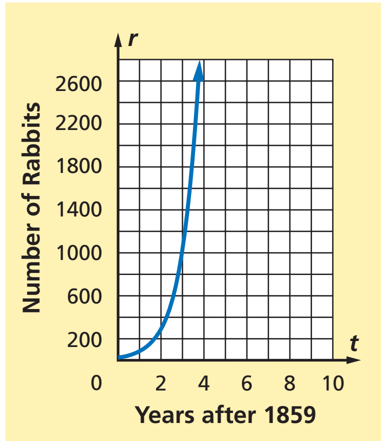
STOP **QY1**

> **QY1**
>
> Using the equation $r = 24(3.5)^t$, estimate the population of rabbits $\frac{1}{2}$ year after their introduction.

In the graph at the left below, $r = 24(3.5)^t$ is plotted for values of t from 0 to 3.75, increasing by 0.25. The middle graph shows values for t from 0 to 3.8, increasing by 0.02.

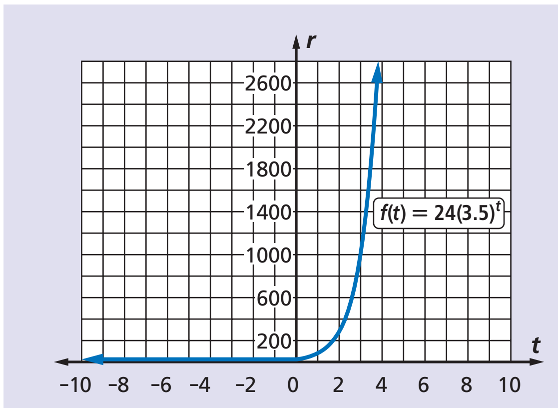






Because time is continuous when measuring population growth, you can think of the function with equation $r = f(t) = 24(3.5)^t$ as being defined for all real nonnegative values of t, as graphed above at the right. However, the equation has meaning for any real number t. Using the set of real numbers as the domain results in the function graphed at the right below.

This graph shows an *exponential curve*. The shape of an exponential curve is different from the shape of a parabola, a hyperbola, or an arc of a circle. The range of the function f is the set of positive real numbers. Its graph never intersects the t-axis, but gets closer and closer to it as t gets smaller and smaller. Thus, the t-axis is a *horizontal asymptote* to the graph. Substituting $t = 0$ into the equation gives an r-intercept of 24. This represents the number of rabbits present when they were first introduced.



Example 1

Use a CAS and the equation $f(t) = 24(3.5)^t$.

a. Estimate the number of rabbits after 2 years and 3 months.

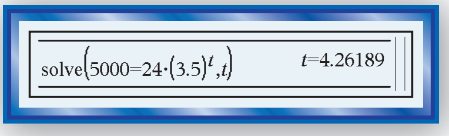
b. Estimate the time to the nearest 0.01 year when there were 5000 rabbits.

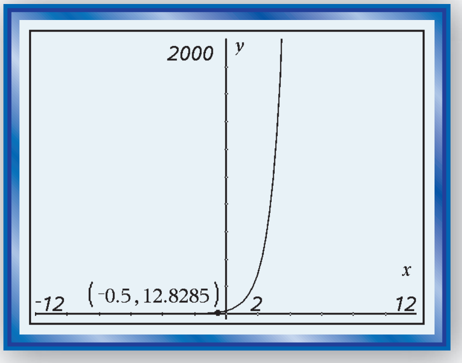
c. Use a graph of f to estimate $f\left(-\frac{1}{2}\right)$.

Solution

a. Two years and 3 months equals 2.25 years. In the formula for $f(t)$, substitute 2.25 for t: $f(2.25) = 24(3.5)^{2.25} \approx 402.13$. There were approximately 400 rabbits 2 years 3 months after the release.

b. There were 5000 rabbits when $f(t) = 5000$. So, solve the equation $5000 = 24(3.5)^t$ for t on a CAS. The population reaches 5000 rabbits after about 4.26 years.





c. Use the window $-12 \le x \le 12$. Since the range of f is the set of positive real numbers, use the window dimension $0 \le y \le 2000$. Use the TRACE feature. $f\left(-\frac{1}{2}\right) \approx 13$

STOP QY2

QY2

According to this model, in what year did the rabbit population reach 21 million, the approximate human population of Australia in 2008?

What Is an Exponential Function?

The equation $f(t) = 24(3.5)^t$ defines a function in which the independent variable t is the exponent, so it is called an *exponential function*.

Definition of Exponential Function

The function f defined by the equation $f(x) = ab^x$ ($a \neq 0$, $b > 0$, $b \neq 1$) is an **exponential function**.

The graph of an exponential function is called an **exponential curve**. This particular exponential curve models *exponential growth*; that is, as time increases, so does the population of rabbits. The accelerating increase in the number of rabbits is typical of exponential growth situations.

In the equation $y = ab^x$, with $a > 0$, b is the **growth factor**; it corresponds to the constant ratio r in a geometric sequence. The rabbit situation involves the growth factor $b = 3.5$. In general, when $b > 1$, exponential growth occurs.

The compound interest formula $A = P(1 + r)^t$, when P and r are fixed, also defines an exponential function of t. In this case, A is the dependent variable and $1 + r$ is the growth factor. Since $r > 0$, $1 + r$ is greater than one, and compound interest yields exponential growth. The table below shows how geometric sequences and compound interest are modeled by exponential functions.

	Formula	Independent Variable	Dependent Variable	Starting Value	Growth Factor
Geometric Sequence	$g_n = g_1 r^{n-1}$	n	g_n	g_0 or g_1 = first term	r
Compound Interest	$A = P(1 + r)^t$	t	A	P = principal	$1 + r$
Exponential Function	$y = ab^x$	x	y	a = y-intercept	b

GUIDED

Example 2

The speed of a supercomputer is measured in *teraflops*, or trillions of "floating point operations" per second. In 2005, the Blue Gene/L supercomputer recorded a speed of 280.6 teraflops. Over the last 30 years, the speed of the fastest supercomputers has been growing at about 78% per year. Suppose that this growth rate continues, and let $C(x)$ = the speed in teraflops of the fastest supercomputer x years after 2005.

Blue Gene/L at Livermore National Laboratory

a. Write a formula for $C(x)$.

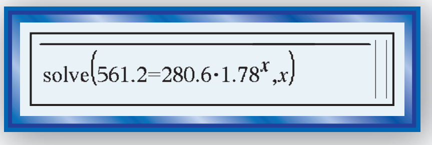
b. Use your formula to predict how long will it take for the fastest supercomputer speed to double the Blue Gene/L record to 561.2 teraflops.

c. Predict how many more years it will take for the fastest supercomputer speed to double again to 1122.4 teraflops.

Solution

a. Model this constant growth situation with an exponential function $C(x) = ab^x$. The initial speed a = ___?___. An annual growth rate of 78% means that each year the computer speed is 178% of the previous year's speed, so b = ___?___. A model is $C(x)$ = ___?___.

b. Solve 561.2 = ___?___(___?___)^x on a CAS to get $x \approx$ ___?___. It will take about ___?___ years for the speed to double.



c. Solve 1122.4 = ___?___. So, $x \approx$ ___?___. Because ___?___ − 1.2 = ___?___, it will take about ___?___ more years for the speed to double a second time.

Parts b and c of Example 2 demonstrate that, with an exponential growth model, the computing speed doubles in the same amount of time regardless of when you start. This constant doubling time is a general feature of exponential growth.

Questions

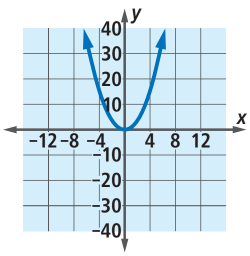
COVERING THE IDEAS

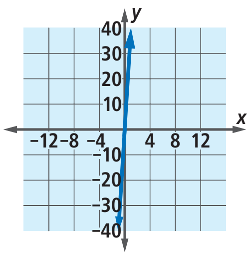
In 1–3, use the rabbit population model from Example 1.

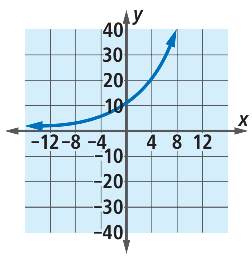
1. About how many rabbits were there 6.2 years after they were introduced to Australia?
2. After about how many years were 100 million rabbits present?
3. Suppose that Thomas Austin had released only 10 rabbits for hunting. At the same annual growth factor of 3.5, about how many rabbits would there then have been after 5 years?
4. Define *exponential function*.
5. **Multiple Choice** Which is an equation for an exponential function?

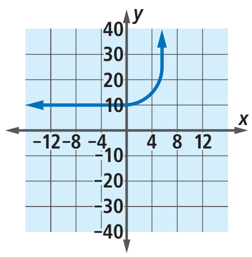
   A $y = x^{3.04}$ B $y = 3.04x$ C $y = 3.04^x$

6. **Multiple Choice** Which graph below best shows exponential growth? Explain your answer.

   A
   

   B
   

   C
   

   D
   

7. Let f be a function with $f(x) = 4 \cdot 2^x$.
   a. Graph $y = f(x)$. b. Approximate $f(-1.4)$.
   c. **True or False** f is an exponential function. Explain.
8. Consider the exponential curve with equation $y = ab^x$, where $b > 1$.
   a. **Fill in the Blank** The y-intercept is __?__.
   b. **Fill in the Blank** The constant growth factor is __?__.
   c. Which line is an asymptote to the graph?

In 9 and 10, refer to the situation and function $C(x) = 280.6(1.78)^x$ from Example 2.

9. a. Find $C(-10)$.
   b. In terms of the situation, what does $C(-10)$ represent?
10. a. Use the model to estimate the computing speed of the fastest supercomputer in the year 2010.
    b. Some researchers believe that a supercomputer capable of 10^4 teraflops could simulate the human brain. If current trends continue, when would such computers be possible?

11. Consider the exponential function with equation $A = P(1 + r)^t$.

    a. Name the independent and dependent variables.

    b. What is the growth factor?

APPLYING THE MATHEMATICS

12. Refer to the function $C(x) = 280.6(1.78)^x$.

    a. Find the average rate of change between $x = -4$ and $x = -2$.

    b. Find the average rate of change between $x = 2$ and $x = 4$.

    c. What conclusions can you draw from your answers in Parts a and b?

13. In 2000, the population of Canada was about 31.1 million people, and the population of Morocco was about 30.2 million people. Over the period 1950–2000, the population of Canada grew at an average rate of about 1.16% annually, while the population of Morocco grew at an average rate of about 2.38% annually. Let the function C with equation $C(x) = a \cdot b^x$ represent the population of Canada x years after 2000, in millions, and let the function M with $M(x) = c \cdot d^x$ represent the population of Morocco x years after 2000, in millions.

    a. Determine the values of a, b, c, and d and write formulas for $C(x)$ and $M(x)$.

    b. Use your formulas from Part a to estimate the populations of Canada and Morocco in 1995.

    c. Graph $y = C(x)$ and $y = M(x)$ on the same set of axes for $0 \le x \le 50$.

    d. Make a prediction comparing the future populations of Canada and Morocco if current trends continue.

Morocco is on the northwest side of Africa and is slightly larger than California.

14. In September of 2004, the online user-edited encyclopedia *Wikipedia* contained about 1,000,000 articles. In January of 2001, the month it was launched, it contained 617 articles. This means it has grown on average about 18% per month since its launch date. Let $f(x) = ab^x$ represent the number of articles on *Wikipedia* x months after January of 2001 ($x = 0$ represents January of 2001).

    a. From the given information, find the values of a and b.

    b. According to this model, how many articles were there in September of 2004? Explain why the answer is not exactly 1,000,000.

    c. According to this model, how many months did it take for the number of *Wikipedia* articles to double from 1,000,000 to 2,000,000?

15. In 1993, a sample of fish caught in a Mississippi River pool on the Missouri-Illinois border included 1 bighead carp. In 2000, a same-size sample from the same pool included 102 such fish. Other samples taken during this period support an exponential growth model. Let $f(x) = ab^x$ represent the number of carp caught x years after 1993.
   a. Use the information given to determine the value of a.
   b. Find an approximate value for b.
   c. According to your model, how many bighead carp would be in a similar sample caught in 2014?

Hypophthalmichthys nobilis, **the bighead carp, is an invasive species of fish that was first introduced in 1986.**

REVIEW

16. Suppose $f(x) = 5x - 6$. **(Lessons 8-3, 8-2)**
   a. Find an equation for f^{-1}.
   b. Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.
   c. **True or False** The graphs in Part b are reflection images of each other.

17. The matrix $\begin{bmatrix} 3 & 3 & -3 \\ -1 & -2 & 3 \end{bmatrix}$ represents triangle TRI. **(Lessons 4-10, 4-1)**
   a. Give the matrix for the image of $\triangle TRI$ under $T_{1,-2}$.
   b. Graph the preimage and image on the same set of axes.

18. Liberty Lumber sells 6-foot long 2-by-4 boards for \$1.70 each, and 8-foot long 2-by-6 boards for \$2.50 each. Last week they sold \$500 worth of these boards. Let x be the number of 2-by-4s sold and y be the number of 2-by-6s sold. **(Lesson 3-2)**
   a. Write an equation relating x and y.
   b. If 200 2-by-4s were sold, how many 2-by-6s were sold?

19. Suppose a new car costs \$28,000 in 2009. Find its value one year later, in 2010, if **(Previous Course)**
   a. the car is worth 82% of its purchase price.
   b. the car depreciated 20% in value.
   c. the value of the car depreciated r%.

EXPLORATION

20. The Australian rabbit plague was initiated by introducing 24 rabbits into the country.
   a. Suppose 12 rabbits, not 24, had been introduced in 1859. How, then, would the number of rabbits in later years have been affected?
   b. Answer Part a if 8 rabbits had been introduced.
   c. Generalize Parts a and b.

QY ANSWERS

1. 45

2. 1869