Lesson 12-9 Quadratic-Quadratic Systems

Vocabulary

quadratic-quadratic system

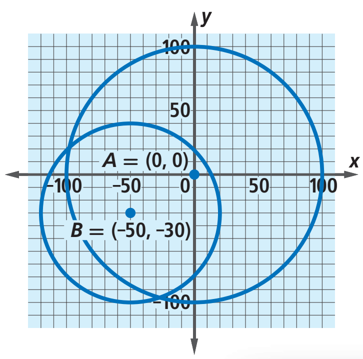
BIG IDEA Solutions to systems with two quadratic equations in *x* and *y* can be found by graphing, using linear combinations, by substitution, or using technology.

Recall from Lesson 12-2 that an earthquake sends out seismic waves from its epicenter. A single monitoring station can determine that the epicenter lies on a particular circle with the station as its center, but it takes multiple stations combining their information to locate the epicenter.

Example 1

An earthquake monitoring station *A* determines that the center of a quake is 100 km away. A second station *B* 50 km west and 30 km south of the first finds that it is 70 km from the quake's center. Find all possible locations of the epicenter in relation to station *A*.

Solution Let Station *A* be located at (0, 0). Then Station *B* is at (–50, –30). Draw the circles with radii 100 and 70. The graph shows two intersections of the circles. These are the possible locations of the epicenter.



Because the epicenter is 100 km from *A*, the larger circle has equation $x^2 + y^2 = 10{,}000$. Because the epicenter is 70 km from *B*, the smaller circle has equation $(x + 50)^2 + (y + 30)^2 = 4900$. These two equations form the following *quadratic-quadratic system*.

$$\begin{cases} x^2 + y^2 = 10{,}000 \\ (x + 50)^2 + (y + 30)^2 = 4900 \end{cases}$$

In order to find the exact solutions to this system, you can use by-hand methods or a CAS. One CAS solution is partially shown at the right. The full solution, found by scrolling the display, is:

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x = -97.7251 and y = 21.2085 or
x = -27.2749 and y = -96.2085
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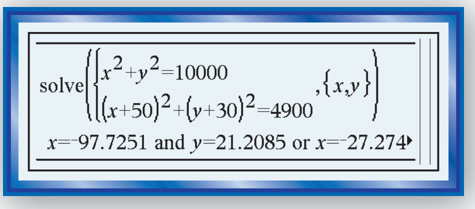


Mental Math

Give an inequality that represents the situation.

a. The price *n* of a gallon of gasoline now is more than 4 times the price *t* of a gallon 10 years ago.

b. The life ℓ of a battery is guaranteed to be at least 30 hours.

c. The average test score *s* was 86.4 with a 0.5-point margin of error.

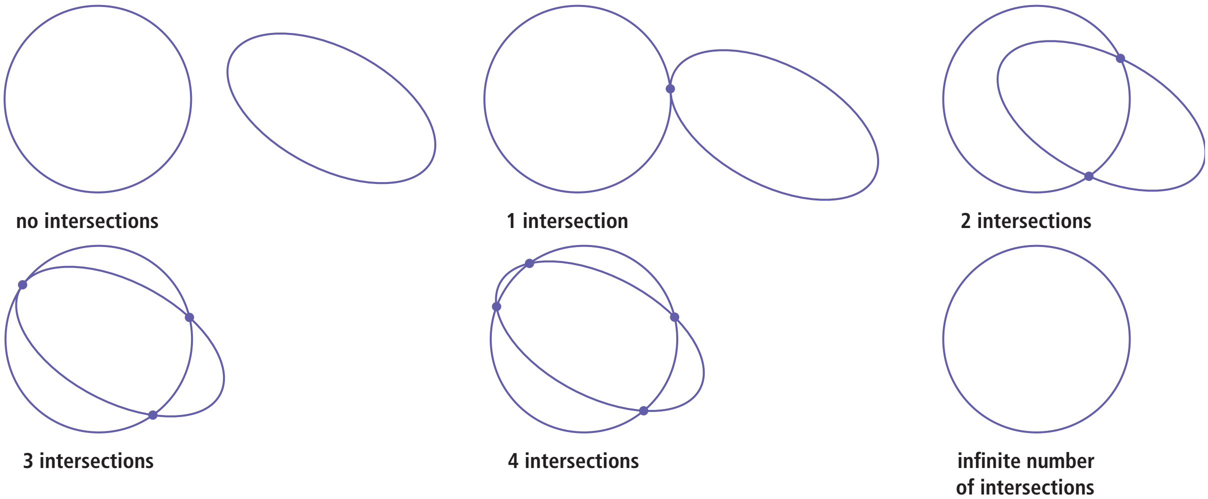
d. You can order from the children's menu if your age *a* is under 7 years.

The epicenter is located either about 98 kilometers west and 21 kilometers north of Station A, or about 27 kilometers west and 96 kilometers south of Station A. Information from a third station would help you determine which was the actual location.

STOP QY

In general, a **quadratic-quadratic system** involves two or more quadratic sentences. Geometrically, a quadratic-quadratic system involves curves represented by quadratic relations: circles, ellipses, hyperbolas, and parabolas.

In Lesson 12-8, you looked at ways a line could intersect the different conic sections. Below are examples of the six different ways that a quadratic-quadratic system of a circle and an ellipse can intersect.



Activity

MATERIALS DGS (optional)

Using the examples above as a guide, fill in a table like the one at the right for the possible numbers of intersections for each quadratic-quadratic system. You may find a DGS helpful.

	Parabola	Circle	Ellipse	Hyperbola
Parabola	?	?	?	?
Circle	?	?	?	?
Ellipse	?	0, 1, 2, 3, 4, or infinitely many	?	?
Hyperbola	?	?	?	?

The results of the Activity can be generalized. Quadratic-quadratic systems may have 0, 1, 2, 3, 4, or infinitely many solutions (if the quadratics are equivalent equations).

Some quadratic-quadratic systems are simple enough to be solved by hand.

▶ **QY**

Check both solutions to Example 1 in both equations of the system.

GUIDED

Example 2

Find all solutions (x, y) to $\begin{cases} 4x^2 + 3y^2 = 37 \\ 3x^2 - y^2 = 5 \end{cases}$.

Solution This system represents the intersection of an ___?___ and a ___?___. There are 0, 1, 2, 3, or 4 possible solutions. Use the linear combination method to find them.

$$4x^2 + 3y^2 = 37$$

$$\underline{\ ?\ }x^2 - \underline{\ ?\ }y^2 = \underline{\ ?\ } \qquad \text{Multiply the second equation by 3.}$$

$$\underline{\ ?\ }x^2 = \underline{\ ?\ } \qquad \text{Add the equations to eliminate } y.$$

$$x^2 = \underline{\ ?\ }$$

$$x = \pm\ \underline{\ ?\ }$$

Substitute each value of x into one of the given equations.

If $x = \underline{\ ?\ }$, then $4(\underline{\ ?\ })^2 + 3y^2 = 37$, so $y = \pm\ \underline{\ ?\ }$.

If $x = \underline{\ ?\ }$, then $4(\underline{\ ?\ })^2 + 3y^2 = 37$, so $y = \pm\ \underline{\ ?\ }$.

So, this system has 4 solutions: $(\underline{\ ?\ }, \underline{\ ?\ })$, $(\underline{\ ?\ }, \underline{\ ?\ })$, $(\underline{\ ?\ }, \underline{\ ?\ })$, and $(\underline{\ ?\ }, \underline{\ ?\ })$.

Check Solve on a CAS.

solve$(4x^2+3y^2=37 \text{ and } 3x^2-y^2=5, x, y)$

The following quadratic-quadratic system involves the intersection of two rectangular hyperbolas.

Example 3

In one month, Willie's Western Wear took in $10,500 from boot sales. Although Willie sold 30 fewer pairs of boots the next month, he still sold $10,800 worth of boots by raising the price $10 per pair. Find the price of a pair of boots in each month.

Civil War era military boots were not the ideal boot for cowboys. So, in the late 1800s, boots were given pointy toes to help feet get into stirrups and higher heels to help feet stay there.

Solution Let n = the number of pairs of boots sold in the first month. Let p = the price of a pair of boots in the first month.

The equations for total sales in the first and second months, respectively, are:

(1) $$np = 10{,}500$$

(2) $$(n - 30)(p + 10) = 10{,}800$$

Our goal is to obtain an equation in only one of these variables.

In (1), solve for p. $\quad p = \frac{10{,}500}{n}$

In (2), expand. $\quad np + 10n - 30p - 11{,}100 = 0$

The two forms of equation (1) allow you to make two substitutions into equation (2). Substitute 10,500 for np and $\frac{10{,}500}{n}$ for p to get an equation in one variable.

$$10{,}500 + 10n - 30\left(\frac{10{,}500}{n}\right) - 11{,}100 = 0.$$

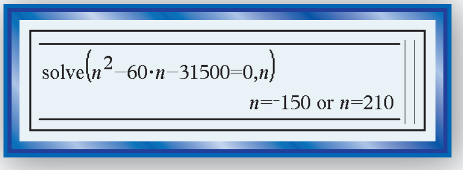
Divide both sides by 10.

$$1050 + n - 3\left(\frac{10{,}500}{n}\right) - 1110 = 0$$

Simplify. $\quad n - 60 - \frac{31{,}500}{n} = 0$

Multiply both sides by n. $\quad n^2 - 60n - 31{,}500 = 0$

This is a quadratic equation you can solve in many ways. One CAS solution is shown at the right.



The number n of pairs of boots sold can only be positive.

So, $\quad n = 210.$

Since the price $p = \frac{10{,}500}{n}$, $p = \frac{10{,}500}{210} = 50.$

The boots were priced at \$50 a pair the first month, and $p + 10 = \$60$ a pair the second month.

Check With these numbers, 210 pairs of boots were sold the first month at \$50/pair, so Willie took in \$10,500. That checks. The second month, Willie sold 180 pairs of boots at \$60/pair. So he took in \$10,800. The solution checks.

Questions

COVERING THE IDEAS

1. A system with two quadratic equations in x and y that are not equivalent can have at most how many solutions?

2. Without doing any algebra or graphing, tell how many solutions the system $x^2 + y^2 = 9$ and $3x^2 + 3y^2 = 48$ has. Explain your response.

3. Find all solutions (x, y) to $\begin{cases} 9x^2 + 4y^2 = 36 \\ -16x^2 + 4y^2 = 36 \end{cases}$.

4. **True or False** The solutions to the system in Example 2 lie on the symmetry lines of both the hyperbola and the ellipse.

5. Consider the circle $x^2 + y^2 = 9$ and the parabola $y = x^2$.
   a. How many intersection points do you expect?
   b. Check your answer to Part a by finding the points of the intersection.

In 6 and 7, refer to Example 3.

6. a. What two substitutions could you make to transform the second equation into an equation in terms of p only?
   b. Check the solution by solving the original system using the substitutions in Part a.

7. In the second month, if Willie had instead raised prices $18 per pair and earned $12,000 from selling 50 fewer pairs of boots than the previous month, what would have been the price of a pair of boots in each month?

APPLYING THE MATHEMATICS

8. The product of two real numbers is 6984. If one number is increased by 4 and the other is decreased by 10, the new product is 6262.
   a. Write a quadratic system that can be solved to find these numbers.
   b. Find the numbers.

In 9–11, solve the system.

9. $\begin{cases} x^2 + y^2 = 25 \\ y = x^2 - 13 \end{cases}$

10. $\begin{cases} \frac{x^2}{16} + \frac{y^2}{9} = 1 \\ x^2 - y^2 = 7 \end{cases}$

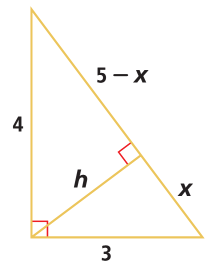
11. $\begin{cases} y = x^2 + 3x - 4 \\ y = 2x^2 + 5x - 3 \end{cases}$

12. Draw a graph of the solution set of the system $\begin{cases} y \geq x^2 \\ x^2 + y^2 \leq 4 \end{cases}$.

13. A fleet of fishing boats locates its nets by sonar. One boat determines that a net is 1000 meters away. A second boat, 200 meters east and 800 meters north of the first, finds that it is 400 meters from the net. A third boat, 1000 meters east and 1100 meters north of the first, finds that it is 500 meters away from the net.

   a. Suppose the first boat is at the origin of a coordinate system. Write a system of three equations to describe this situation.
   b. The graphs of the three equations have one intersection point in common. Solve the system and find the location of the net with respect to the first boat.

14. The altitude h of the 3-4-5 triangle at the right splits the hypotenuse into segments of length x and $5 - x$.



   a. Use the Pythagorean Theorem to create two quadratic relations in terms of x and h.
   b. Solve your system of equations in Part a for x and h.
   c. Which solution(s) make(s) sense in the context of this problem?

REVIEW

15. The sum of Mr. Hwan's age in years and his baby's age in months is 37. The product of the ages is 300. Solve a system of equations to find their ages. (Lesson 12-8)

In 16 and 17, an equation is given. Does the equation represent a quadratic relation? If so, put the equation in standard form for a quadratic relation. If not, explain why not. (Lesson 12-7)

16. $x^2 + 5xy^2 = 10$

17. $\frac{1}{2}x - 13y^2 = \sqrt{7}y$

In 18 and 19, graph the equation. (Lesson 12-6, 12-4)

18. $\frac{x^2}{9} + \frac{y^2}{36} = 1$

19. $\frac{x^2}{9} - \frac{y^2}{36} = 1$

20. The third and fourth terms of a geometric sequence are 20 and –40. (Lesson 7-5)
   a. What is the constant ratio?
   b. What are the first and second terms?
   c. Write an explicit formula for the nth term.
   d. What is the 15th term?

21. Explain why every transformation is a function. (Lesson 4-7)

In 22 and 23, multiply the matrices. (Lesson 4-3)

22. $\begin{bmatrix} 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 0 & 5 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \\ -3 & 4 \end{bmatrix}$

EXPLORATION

24. In 2002, a new method of code breaking called the XSL (eXtended Sparse Linearization) attack was published. The attack is a type of *algebraic cryptanalysis*. Search the Internet to answer these questions.
   a. What is the purpose of the XSL attack?
   b. How does this system use quadratic systems?

QY ANSWER

$(-97.7)^2 + (21.2)^2 = 9994.73 \approx 10{,}000$;
$(-97.7 + 50)^2 + (21.2 + 30)^2 = 4896.73 \approx 4900$;
$(-27.3)^2 + (-96.2)^2 = 9999.73 \approx 10{,}000$;
$(-27.3 + 50)^2 + (-96.2 + 30)^2 = 4897.73 \approx 4900$