Lesson 12-8 Quadratic-Linear Systems

Vocabulary

quadratic system

quadratic-linear system

BIG IDEA Solutions to systems with one quadratic and one linear equation in *x* and *y* can be found by graphing, substitution, using linear combinations, or using a CAS.

A **quadratic system** is a system that involves polynomial sentences of degrees 1 and 2, at least one of which is a quadratic sentence. One way to solve a quadratic system is to examine the points of intersection of the graphs of the equations.

Mental Math

Give all points of intersection of the graph of $y = x^2$ and the graph of the given equation.

a. $y = 0$

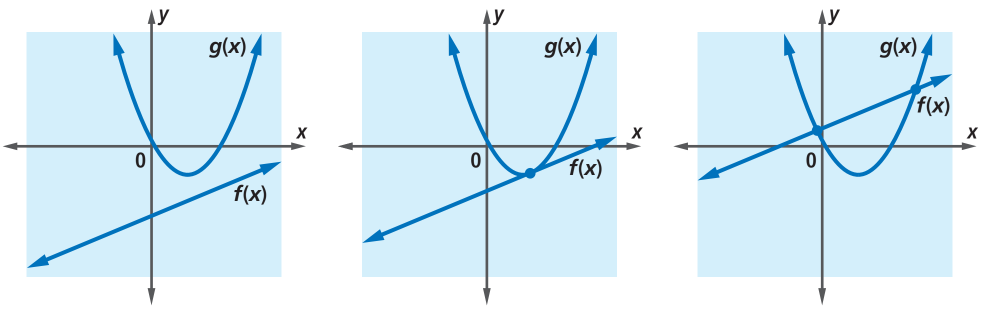
b. $y = 4$

c. $y = x$

d. $y = -1.5$

Examining Quadratic-Linear Systems Geometrically

A quadratic system with at least one linear sentence is called a **quadratic-linear system**. No new properties are needed to solve quadratic-linear systems. Geometrically, the task is to find the intersection of a conic section and a line. For example, in a system of a parabola and a line, there are three possibilities.



0 intersections no solution | 1 intersection one solution | 2 intersections two solutions

The following Activity will help you determine the possible numbers of intersection points of a line and the other quadratic relations that you have studied.

Activity

Step 1 Sketch a circle, an ellipse, and a hyperbola on three separate sets of axes.

Step 2 On the graph of the circle, sketch several lines that intersect the circle in various ways. In how many points can a circle and a line intersect?

Step 3 Repeat Step 2 for the ellipse and the hyperbola.

Step 4 Fill in a table like the one at the right with the number of intersections between a line and the indicated quadratic relation. The Line-Parabola cell has been filled in for you.

Quadratic Relation	Number of Intersections with Line
Parabola	0, 1, or 2
Circle	?
Ellipse	?
Hyperbola	?

As you saw in the Activity, a quadratic-linear system may have 0, 1, or 2 solutions. This is because a quadratic equation may have 0, 1, or 2 solutions.

Solving Quadratic-Linear Systems Algebraically

One way to solve a quadratic-linear system is to solve the linear equation for one variable and substitute the resulting expression into the quadratic equation.

Example 1

Find exact solutions to the system $\begin{cases} y + 4x = 10 \\ xy = 4 \end{cases}$.

Solution 1 Geometrically, the solution to this system is the intersection of a line and a hyperbola.

Solve the first sentence for y.

$$y = 10 - 4x$$

Substitute the expression $10 - 4x$ for y in the second sentence.

$$x(10 - 4x) = 4$$

This is a quadratic equation that you can solve by the Quadratic Formula or by factoring. To use the Quadratic Formula, expand the left side and rearrange the equation so one side is zero.

$$10x - 4x^2 = 4$$

$$4x^2 - 10x + 4 = 0$$

$$x = \frac{(-10) \pm \sqrt{100 - 4 \cdot 4 \cdot 4}}{8} = \frac{10 \pm \sqrt{36}}{8}$$

$$x = \frac{10 - 6}{8} \quad \text{or} \quad x = \frac{10 + 6}{8}$$

$$x = \frac{1}{2} \quad \text{or} \quad x = 2$$

(continued on next page)

Now substitute each value into $y = 10 - 4x$ to find y.

When $x = \frac{1}{2}$, $y = 10 - 4\left(\frac{1}{2}\right) = 8$. So, one solution is $\left(\frac{1}{2}, 8\right)$.

When $x = 2$, $y = 10 - 4(2) = 2$. The other solution is $(2, 2)$.

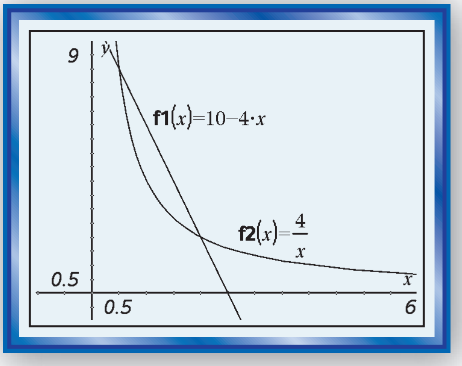
So, the solutions are $\left(\frac{1}{2}, 8\right)$ and $(2, 2)$.

Check Solve both equations for y.

$$y = 10 - 4x \qquad \text{and} \qquad y = \frac{4}{x}$$

Now graph the system with a graphing utility. Zoom in to estimate the coordinates of the intersection points.

The curves intersect at (0.5, 8) and (2, 2). It checks.

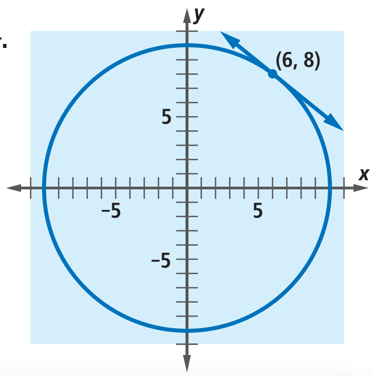


You could also solve Example 1 by using a graphing utility to find the exact coordinates of the intersection points. (See Question 2.)

Example 2 illustrates two solution methods that are quite different from each other. The first solution is algebraic and uses a CAS. The second solution draws on what you have already learned about circles in your study of geometry.

Example 2

At the right are graphs of the equations $x^2 + y^2 = 100$ and $y = -\frac{3}{4}x + \frac{25}{2}$. It appears that they intersect in only one point, (6, 8). (That is, the line is tangent to the circle.) Is this so? Justify your answer.



Solution 1 First, check that the graphs intersect at (6, 8).

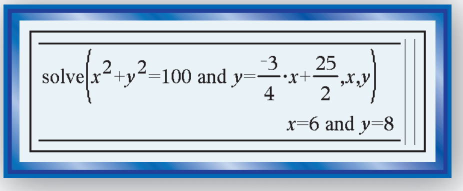
Does $6^2 + 8^2 = 100$? Yes.

Does $8 = -\frac{3}{4}(6) + \frac{25}{2}$? Yes, because $-\frac{3}{4}(6) + \frac{25}{2} = -\frac{9}{2} + \frac{25}{2} = \frac{16}{2} = 8$.

To find all solutions, solve the system $\begin{cases} x^2 + y^2 = 100 \\ y = -\frac{3}{4}x + \frac{25}{2} \end{cases}$.

Use the solve command on a CAS.

The only solution is $x = 6$ and $y = 8$. So, the only point of intersection is (6, 8).



Solution 2 You may remember from your study of geometry that a tangent to a circle is the line perpendicular to the radius at the point of intersection. In this case, a radius from (0, 0) to (6, 8) has slope $\frac{4}{3}$. The line $y = -\frac{3}{4}x + \frac{25}{2}$ has slope $-\frac{3}{4}$. Since the product of the slopes is -1, this line is perpendicular to the radius at (6, 8) and is a tangent.

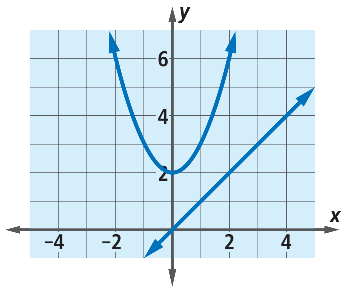
Inconsistent Quadratic Systems

Like linear systems, quadratic systems can be inconsistent. That is, there can be no solution. One signal for inconsistency is that the solutions to the quadratic system are not real. This means that the graphs of the equations in the system do not intersect.

GUIDED

Example 3

Find the points of intersection of the line $y = x$ and the parabola $y = x^2 + 2$.



Solution 1 Graph the line and parabola, as shown at the right.
There are __?__ points of intersection.

Solution 2 Solve the system $\begin{cases} y = x \\ y = x^2 + 2 \end{cases}$.

Substitute x for y in the second sentence.

__?__ $= x^2 + 2$

Put this equation in standard form so you can use the Quadratic Formula.

__?__ − __?__ + __?__ = 0

From that formula, $x = \frac{? \pm \sqrt{?}}{?}$.

Both solutions to this equation are nonreal complex numbers.
So there are __?__ points of intersection.

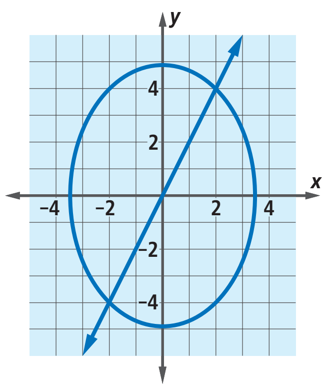
Questions

COVERING THE IDEAS

1. How many solutions can a system of one linear and one quadratic equation have?

2. Refer to Example 1. Check the solutions by graphing the system on a graphing utility and finding the exact coordinates of the points of intersection.

3. A graph of the system $\begin{cases} y = 2x \\ 4x^2 + 2y^2 = 48 \end{cases}$ is shown at the right.



   a. How many solutions are there?
   b. Use the graph to approximate the solutions.
   c. Check your answers in Part b.

In 4 and 5, a system is given. Estimate the solutions by graphing. Then find exact solutions by substitution.

4. $\begin{cases} mn = 12 \\ n = -\frac{2}{3}m + 6 \end{cases}$

5. $\begin{cases} y = 2x^2 + 5x - 1 \\ y = x + 5 \end{cases}$

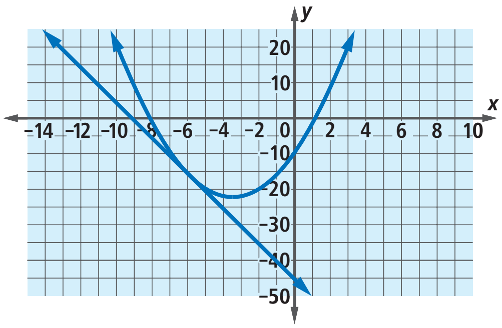
In 6 and 7, find all intersection points of the line and the parabola.

6. $y = x + 4$ and $y = x^2$

7. $y = 2x^2 - 4$ and $y = x - 3$

8. **a.** What term is used to describe a system that has no solutions?
   b. Give equations for such a system involving an ellipse and a line.

In 9 and 10, consider the figure at the right, which suggests that the parabola $y = x^2 + 7x - 10$ and the line $y = -5x - 46$ intersect at or near the point $(-6, -16)$.



9. Check by substitution that this point is on both curves.

10. Solve the system algebraically to verify that this is the only solution.

APPLYING THE MATHEMATICS

11. Explain why a quadratic-linear system cannot have an infinite number of solutions.

12. Phillip has 200 meters of fencing material. Next to his barn, he wants to fence in a rectangular region with area 1700 square meters, using a wall of the barn as one of the sides of the enclosure.

   a. Let x be the width of the region and y be its length. Draw a picture of the situation.
   b. Assuming Phillip uses all 200 meters of fencing, write a system of equations that can be solved to find x and y.
   c. Graph your system to estimate the dimensions of this region.
   d. Solve your system from Part b and explain your answer in the context of the problem.

13. Solve a quadratic-linear system and use the solution to explain why it is impossible to have two real numbers whose sum is 8 and whose product is 24.

In 14 and 15, solve and check.

14. $\begin{cases} 4r^2 + s^2 = 9 \\ 3r + s = 1 \end{cases}$

15. $\begin{cases} \frac{x^2}{16} + \frac{y^2}{36} = 1 \\ y = \frac{1}{2}x \end{cases}$

REVIEW

16. Consider the hyperbola with equation $xy = k$, where $k > 0$. (Lesson 12-7)
    a. Give the coordinates of its foci.
    b. Identify its asymptotes.
    c. Determine the focal constant.

17. Find an equation for the hyperbola with foci at (10, 10) and (–10, –10) and focal constant 20. (Lesson 12-7)

18. Give an equation for a hyperbola that
    a. is the graph of a function.
    b. is not the graph of a function. (Lessons 12-7, 12-6, 1-4)

In 19–21, a polynomial is given.
a. Tell whether the polynomial is a binomial square, a difference of squares, or a sum of squares.
b. Factor, if possible, over the integers. (Lesson 11-3)

19. $r^{20} - t^2$
20. $49a^2 - 42ab + 9b^2$
21. $x^2 + 2500$

In 22 and 23, simplify without a calculator. (Lessons 8-7, 7-8, 6-9)

22. $\left(2 - \sqrt{-9}\,\right)\left(2 + \sqrt{-9}\,\right)$
23. $\sqrt{-64} + \sqrt[3]{-64} + 64^{-\frac{1}{6}}$

24. Suppose a car rental company charges a flat rate of \$25 plus 45 cents for each mile or part of a mile driven. (Lesson 3-9)
    a. Write an equation relating the total charge c to the number m of miles driven.
    b. How much would you pay if you drove 36.3 miles in a rental car?
    c. The company also allows you to pay a flat rate of \$65 a day for an unlimited number of miles. How many miles would you have to drive in a day to make this the better deal?

The first U.S. airport location of a car rental company opened at Chicago's Midway Airport in 1932.

EXPLORATION

25. Find equations for a noncircular ellipse and an oblique (slanted) line that have exactly one point of intersection.