Lesson

12-7 A General Equation for Quadratic Relations

Vocabulary

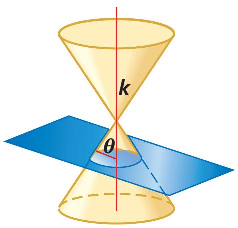
conic section

standard form of an equation for a quadratic relation

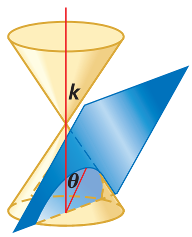
BIG IDEA Equations for specific parabolas, ellipses, circles, and hyperbolas are all special cases of one general equation form.

The Conic Sections

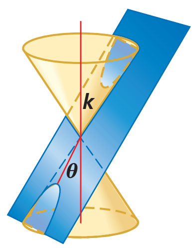
On the opening page of this chapter, you read that parabolas, hyperbolas, and ellipses can all be formed when a plane and a double cone intersect. A double cone is formed by rotating a line (in space) about a line it intersects (its axis). Any of the possible images of the line is an *edge* of the cone. The intersection of a plane and a double cone is called a **conic section**. Let k be the measure of the acute angle between the axis of the double cone and its edge. Let θ be the measure of the smallest angle between the axis and the intersecting plane. The following three possible relationships between θ and k determine the three types of conic sections.



$k < \theta < 90°$
ellipse



$\theta = k$
parabola



$\theta < k$
hyperbola

Mental Math

Serena is cutting a pie into 8 wedge-shaped pieces.

a. If she wants all the pieces to be the same size, what is the measure in degrees of the angle of each wedge?

b. If she wants each of four of the pieces to be twice as big as each of the other four, what is the measure in degrees of the angle of the larger wedges?

c. If she wants each of two of the pieces to be twice as big as each of the other six, what is the measure in degrees of the angle of the larger wedges?

The Standard Form of an Equation for a Quadratic Relation

You have now seen equations for all three different types of conics. Here are some of these equations in standard form, including the hyperbola form $xy = k$ that you will examine in this lesson.

$y = ax^2 + bx + c$	parabola
$(x - h)^2 + (y - k)^2 = r^2$	circle (special ellipse)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $xy = k$	hyperbola

Although the equations for a hyperbola and an ellipse look similar, the others look different. However, all these equations contain only constant terms or terms with x^2, xy, y^2, x, or y. Thus, all the conic sections are special types of relations with polynomial equations of 2nd degree. These are called *quadratic relations*. The **standard form of an equation for a quadratic relation** is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, and F are real numbers, and at least one of A, B, or C is nonzero.

Example 1

Show that the circle with equation $(x - 5)^2 + (y - 1)^2 = 12$ is a quadratic relation.

Solution Rewrite the equation in the standard form of an equation for a quadratic relation. First expand the squares of the binomials and combine like terms.

$$x^2 - 10x + y^2 - 2y + 26 = 12$$

Then add –12 to both sides and use the Commutative Property of Addition to reorder the terms so that they are in the order x^2, xy, y^2, x, y, and constants.

$$x^2 + 0xy + y^2 - 10x - 2y + 14 = 0$$

This is in standard form with $A = 1$, $B = 0$, $C = 1$, $D = -10$, $E = -2$, and $F = 14$. Because at least one of A, B, or C is nonzero, this is a quadratic relation.

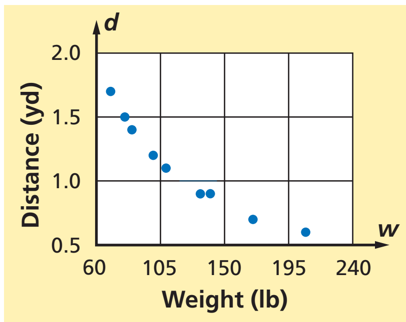
STOP QY1

QY1

If you put the equation $xy = 25$ for a hyperbola into standard form, what are the values of A, B, C, D, E, and F?

Generating More Hyperbolas from Inverse Variation

Recall Anna and Jenna Lyzer's experiment to verify the Law of the Lever in Lesson 2-7. With Anna seated at a fixed point on one side of a seesaw's pivot, her friends with weights w took turns balancing her while Jenna recorded their distances d feet from the pivot. Anna and Jenna found that the inverse-variation function with equation $d = \frac{119}{w}$ is a good model.

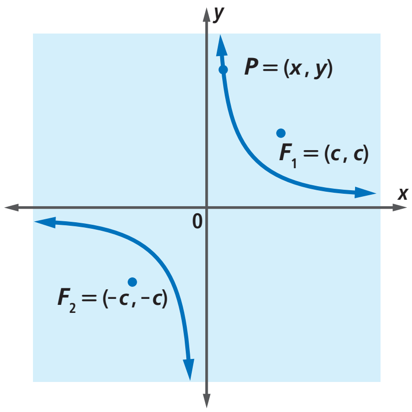


The Lyzers' equation is one instance of the general inverse-variation function, $y = \frac{k}{x}$. In Chapter 2 we claimed that the graph of $y = \frac{k}{x}$ is a hyperbola. Now you can prove it by showing that it satisfies the geometric definition of hyperbola given in the last lesson.

If $k > 0$, each branch of the graph is reflection-symmetric over the line $y = x$, and so the foci must be on the line $y = x$. Because the graph of $y = \frac{k}{x}$ is rotation-symmetric about the origin, the foci are also rotation-symmetric about the origin. Example 2 shows how an equation of the form $y = \frac{k}{x}$ arises when the foci meet these criteria.

Example 2

Find an equation of the form $y = \frac{k}{x}$ for the hyperbola with foci $F_1 = (c, c)$ and $F_2 = (-c, -c)$ and focal constant $2c$.



Solution Let $P = (x, y)$ be a point on the hyperbola. Then, by the definition of hyperbola, one branch of the curve is the set of points P such that

$$PF_1 - PF_2 = 2c.$$

Use the Distance Formula with $F_1 = (c, c)$, and $F_2 = (-c, -c)$.

$$\sqrt{(x-c)^2 + (y-c)^2} - \sqrt{(x+c)^2 + (y+c)^2} = 2c$$

Now proceed as in Lesson 12-4 with the derivation of an equation for an ellipse. You may find your CAS helpful as you follow the steps. First add $\sqrt{(x+c)^2 + (y+c)^2}$ to both sides.

$$\sqrt{(x-c)^2 + (y-c)^2} = 2c + \sqrt{(x+c)^2 + (y+c)^2}$$

Square both sides. Notice that the right side is a binomial.

$$(x-c)^2 + (y-c)^2 = 4c^2 + 4c\sqrt{(x+c)^2 + (y+c)^2} + (x+c)^2 + (y+c)^2$$

Expand the squares of the binomials, combine like terms, and simplify.

$$-4cx - 4cy - 4c^2 = 4c\sqrt{(x+c)^2 + (y+c)^2}$$

Divide by $-4c$; then square both sides.

$$(x + y + c)^2 = (x+c)^2 + (y+c)^2$$

The other branch of the hyperbola also satisfies this equation. Expand both sides again and simplify.

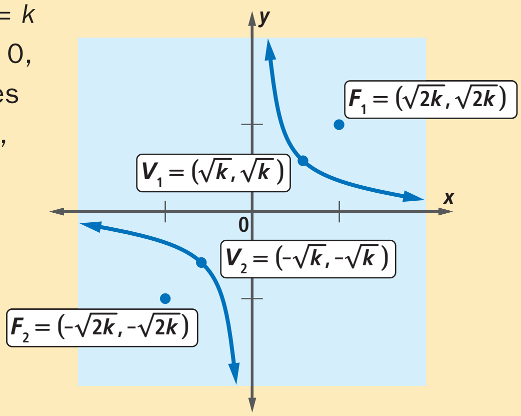
$$x^2 + 2xy + 2cx + y^2 + 2cy + c^2 = x^2 + 2cx + c^2 + y^2 + 2cy + c^2$$
$$2xy = c^2$$
$$xy = \frac{c^2}{2}$$
$$y = \frac{\frac{c^2}{2}}{x}$$

This is an equation of the form $y = \frac{k}{x}$, where $k = \frac{c^2}{2}$.

The following theorem summarizes attributes of hyperbolas of the form $xy = k$.

Attributes of $y = \frac{k}{x}$ Theorem

The graph of $y = \frac{k}{x}$ or $xy = k$ is a hyperbola. When $k > 0$, this hyperbola has vertices $(\sqrt{k}, \sqrt{k})$ and $(-\sqrt{k}, -\sqrt{k})$, foci $(\sqrt{2k}, \sqrt{2k})$ and $(-\sqrt{2k}, -\sqrt{2k})$, and focal constant $2\sqrt{2k}$. The asymptotes of the graph are $x = 0$ and $y = 0$.



STOP **QY2**

> **QY2**
> According to the theorem above, what are the foci and focal constant of the hyperbola that is the graph of $d = \frac{160}{w}$?

GUIDED

Example 3

Find an equation of the form $y = \frac{k}{x}$ for the hyperbola with foci $F_1 = (8, 8)$ and $F_2 = (-8, -8)$, and focal constant 16.

Solution From the Attributes of $y = \frac{k}{x}$ Theorem, you know that the hyperbola has foci $(\sqrt{2k}, \sqrt{2k})$ and (_?_, _?_).
So (_?_, _?_) $= (\sqrt{2k}, \sqrt{2k})$. Solve for k.

$$\underline{\ ?\ } = \sqrt{2k}$$
$$\underline{\ ?\ } = 2k$$
$$\underline{\ ?\ } = k$$

Substitute _?_ for k in $y = \frac{k}{x}$.

An equation for the hyperbola is $y =$ _?_.

Questions

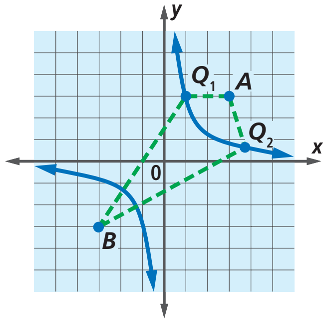
COVERING THE IDEAS

In 1–4, determine whether the equation is for a quadratic relation. If so, put the equation in standard form of an equation for a quadratic relation. If not, tell why not.

1. $(x - 3)^2 + (y + 7)^2 = 118$
2. $2x^2 + 4x - 7xy + 8y^2 + 3y = -3$
3. $\pi y - 6xy + y^2 + x^2 = \sqrt{11}$
4. $-2x^2y + 2xy + 2x^2 + 6x + 5y = 0$

5. Recall Anna and Jenna's equation, $d = \frac{119}{w}$.
   a. What are the foci of the hyperbola with this equation?
   b. What is the focal constant?

6. At the right is a hyperbola with foci A and B. What must be true about $|Q_1A - Q_1B|$ and $|Q_2A - Q_2B|$?



7. The graph of $dw = 240$ is a hyperbola. What are the asymptotes of this hyperbola?

8. Consider the hyperbola with equation $xy = k$ where $k > 0$. Name
   a. its foci.
   b. its asymptotes.
   c. its focal constant.
   d. its vertices.

9. Graph the hyperbola with equation $xy = 8$. Name its foci, vertices, asymptotes, and focal constant.

10. a. Find an equation for the hyperbola with foci at (15, 15) and (–15, –15) and focal constant 30.
    b. Verify that the point (5, 22.5) is on the hyperbola in Part a.

APPLYING THE MATHEMATICS

In 11 and 12, rewrite the equation in the standard form for a quadratic relation, and give the values of *A*, *B*, *C*, *D*, *E*, and *F*.

11. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

12. $y = 5(x + 2)^2 - 11$

13. A hyperbola with perpendicular asymptotes is called a *rectangular hyperbola*. Explain whether or not each of the following is an equation for a rectangular hyperbola.
    a. $xy = 32$
    b. $x^2 - y^2 = 1$
    c. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

14. A car travels the 2.5 miles around the Indianapolis Motor Speedway in t seconds at an average rate of r mph. Racing fans with stopwatches can calculate how fast the car is traveling if they know the value of the constant rt.
    a. What is that value? (*Hint:* Convert miles per second to miles per hour.)
    b. Graph $rt = k$, where k is your answer to Part a.
    c. If a driver completes one lap in 45 seconds, how fast is the driver traveling in mph?

The Indianapolis Motor Speedway is the largest stadium in the United States. It seats 250,000 people.

15. Sketch a graph of the inequality.
    a. $xy > 4$
    b. $xy \leq 4$

REVIEW

16. Consider the hyperbola with the equation $\frac{x^2}{64} - \frac{y^2}{121} = 1$.

a. What are its foci? **b.** Name its vertices.

c. State equations for its asymptotes. **(Lesson 12-6)**

Multiple Choice In 17–19, choose the set of points that best meets the given condition. **(Lessons 12-6, 12-4, 12-2, 12-1)**

A circle **B** ellipse **C** parabola **D** hyperbola

17. equidistant from a given focus and directrix

18. satisfy the equation

$$\left|\sqrt{(x+23)^2 + (y-15)^2} - \sqrt{(x-23)^2 + (y-15)^2}\right| = 7$$

19. satisfy the equation $9x^2 + 2y^2 = 71$

20. Multiple Choice Which of the following describes the set of points P whose distances from (7, 2) and (3, 4) add up to 12? **(Lesson 12-4)**

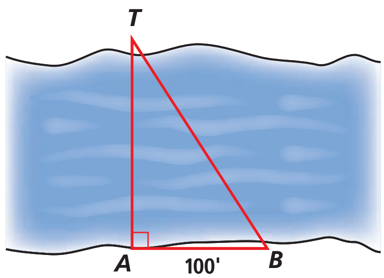
A $(x-7)^2 + (y-2)^2 + (x-3)^2 + (y-4)^2 = 12$

B $(x+7)^2 + (y+2)^2 + (x+3)^2 + (y+4)^2 = 12$

C $\sqrt{(x-7)^2 + (y-2)^2} + \sqrt{(x-3)^2 + (y-4)^2} = 12$

D $\sqrt{(x-7)^2 + (y+2)^2} + \sqrt{5(x+3)^2 + (y+4)^2} = 12$

21. To estimate the distance across a river, Sir Vayer marks point A near one bank, sights a tree at point T growing on the opposite bank, and measures off a distance AB of 100 feet along the bank. At B he sights T again. If $m\angle A = 90°$ and $m\angle B = 76°$, how wide is the river? **(Lesson 10-1)**



22. a. Evaluate $\log_5 125$ and $\log_{125} 5$ without a calculator.

b. Evaluate $\log_4 16$ and $\log_{16} 4$ without a calculator.

c. Generalize Parts a and b. **(Lesson 9-7)**

23. At the zoo, Nigel bought 3 slices of vegetable pizza and 1 small lemonade for \$5.40. Rosa paid \$4.80 for 2 slices of vegetable pizza and 2 small lemonades. What is the cost of a small lemonade? **(Lesson 5-4)**

EXPLORATION

24. In the standard form of a quadratic relation equation, there are six coefficients: A, B, C, D, E, and F. Changing these coefficients affects the type and appearance of the conic section. Search the Internet to find interactive websites that allow you to change the appearance of the conics by varying the coefficients. Make at least three conjectures about how certain types of coefficient changes affect the appearance of the graphs.

QY ANSWERS

1. $A = 0, B = 1, C = 0, D = 0, E = 0, F = -25$

2. The foci are $(\sqrt{320}, \sqrt{320}) = (8\sqrt{5}, 8\sqrt{5})$ and $(-\sqrt{320}, -\sqrt{320}) = (-8\sqrt{5}, -8\sqrt{5})$. The focal constant is $2\sqrt{320} = 16\sqrt{5}$.