

## Lesson

## 12-6

Equations for  
Some Hyperbolas

► **BIG IDEA** From the geometric definition of a hyperbola, an equation for any *hyperbola* symmetric to the  $x$ - and  $y$ -axes can be found.

The edges of the silhouettes of each of the towers pictured at the right are parts of *hyperbolas*. Structures with this shape are able to withstand higher winds and require less material to build than any other form.



### What Is a Hyperbola?

Like an ellipse, a hyperbola is determined by two foci and a focal constant. However, instead of a constant sum of distances from the foci, a point on a hyperbola must be at a *constant difference* of distances from the foci. The following Activity shows one way to find points on a hyperbola.

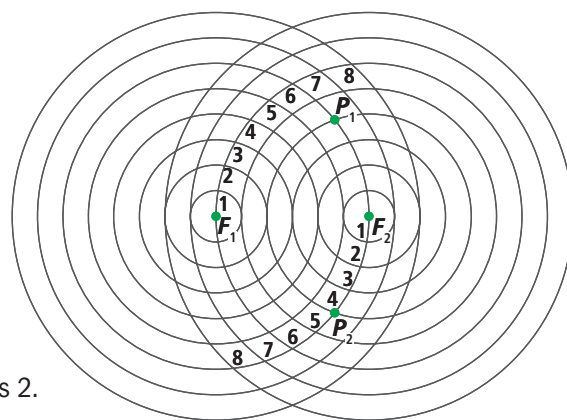
#### Activity

**MATERIALS** conic graph paper with 6 units between the centers of the circles

**Step 1** Copy the foci and points  $P_1$  and  $P_2$  at the right. Find  $P_1F_1$ ,  $P_2F_1$ ,  $P_1F_2$ , and  $P_2F_2$ , then calculate  $P_1F_1 - P_1F_2$  and  $P_2F_1 - P_2F_2$ . Do both differences equal the same constant?

**Step 2** Plot two more points  $P_n$  such that  $P_nF_1 = 8$  and  $P_nF_2 = 6$ , and then two more such that  $P_nF_1 = 7$  and  $P_nF_2 = 5$ . Continue this process to find four more points such that  $P_nF_1 - P_nF_2$  is always 2.

(continued on next page)



### Vocabulary

hyperbola  
foci, focal constant of a hyperbola  
vertices of a hyperbola  
standard position of a hyperbola  
standard form of an equation for a hyperbola

### Mental Math

Suppose a function  $f$  contains the points  $(4, 17)$ ,  $(9, 12)$ , and  $(13, 13)$ .

- Find the rate of change from  $(4, 17)$  to  $(9, 12)$ .
- Find the rate of change from  $(9, 12)$  to  $(13, 13)$ .
- Could the graph of  $f$  be a line?

**Step 3** Repeat Step 2, plotting ten points  $P_n$  such that  $P_nF_2 - P_nF_1 = 2$ .

**Step 4** Draw a smooth curve through the points you plotted in Step 2, and another through the points you plotted in Step 3. These are two branches of a hyperbola. The branches do not intersect.

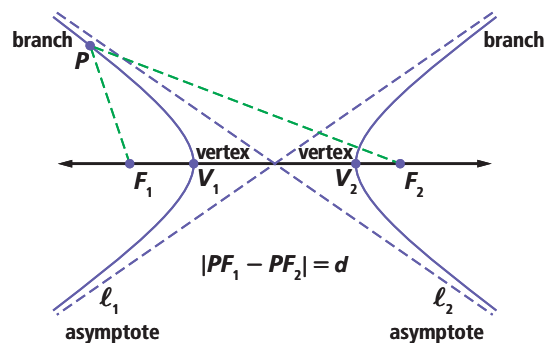
In general, if  $d$  is a positive number less than  $F_1F_2$ , the set of all points  $P$  such that  $|PF_1 - PF_2| = d$  is a hyperbola. The absolute value means that the hyperbola has two branches, one from  $PF_1 - PF_2 = d$ , and the other from  $PF_1 - PF_2 = -d$ . The absolute value function allows both branches to be described with one equation.

### Definition of Hyperbola

Let  $F_1$  and  $F_2$  be any two points and  $d$  be a constant with  $0 < d < F_1F_2$ . Then the **hyperbola** with **foci**  $F_1$  and  $F_2$  and **focal constant**  $d$  is the set of points  $P$  in a plane that satisfy  $|PF_1 - PF_2| = d$ .

The **vertices**  $V_1$  and  $V_2$  of the hyperbola are the intersection points of  $\overleftrightarrow{F_1F_2}$  and the hyperbola.

While it may look like each branch of the hyperbola is a parabola, this is not the case. Each branch of a hyperbola has *asymptotes*. In the figure at the right,  $l_1$  and  $l_2$  are asymptotes. The farther points on the hyperbola are from a vertex of the hyperbola, the closer they are to an asymptote, without ever touching. In contrast, parabolas do not have asymptotes.



## The Standard Form of an Equation for a Hyperbola

A hyperbola is in **standard position** if it is centered at the origin with its foci on an axis. An equation for a hyperbola in standard position resembles the standard form of an equation for an ellipse.

### Equation for a Hyperbola Theorem

The hyperbola with foci  $(c, 0)$  and  $(-c, 0)$  and focal constant  $2a$  has equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , where  $b^2 = c^2 - a^2$ .

**Proof** The proof is almost identical to the proof of the Equation for an Ellipse Theorem in Lesson 12-4. Let  $P = (x, y)$  be any point on the hyperbola with foci  $F_1 = (-c, 0)$  and  $F_2 = (c, 0)$  and focal constant  $2a$ . Then, by the definition of a hyperbola,

$$|PF_1 - PF_2| = 2a.$$

By the definition of absolute value, you know that this equation is equivalent to

$$PF_1 - PF_2 = \pm 2a.$$

Now substitute  $P = (x, y)$ ,  $F_1 = (-c, 0)$ , and  $F_2 = (c, 0)$  into the Pythagorean Distance Formula to get

$$\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = \pm 2a.$$

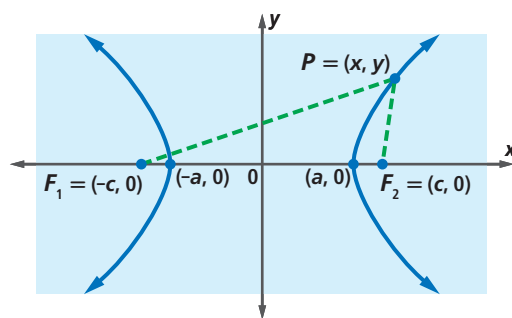
Do algebraic manipulations similar to those in Steps 1–9 of the proof in Lesson 12-4, and the same equation in Step 9 results.

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Then in Step 10, for hyperbolas,  $c > a > 0$ , so  $c^2 > a^2$ . Thus,  $c^2 - a^2$  is positive and you can let  $b^2 = c^2 - a^2$ . So  $-b^2 = a^2 - c^2$ . This accounts for the minus sign in the equation.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is the **standard form of an equation for a hyperbola**.



### Example 1

Find an equation for the hyperbola with foci  $F_1$  and  $F_2$ , where  $F_1F_2 = 10$  and  $|PF_1 - PF_2| = 8$ , on a rectangular coordinate system in standard position.

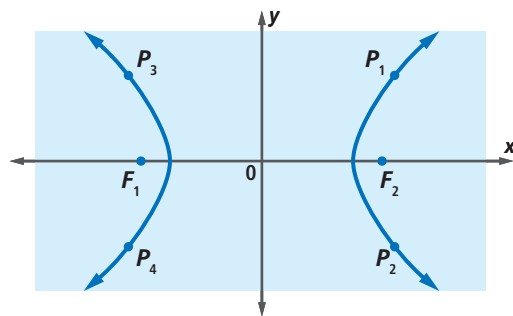
**Solution** Use the Equation for a Hyperbola Theorem.

You are given  $F_1F_2 = 10$ , so  $2c = 10$ , and  $c = 5$ .

The focal constant is 8, so  $2a = 8$ , and  $a = 4$ .

Now,  $b^2 = 5^2 - 4^2 = 9$ . Thus, an equation for

this hyperbola is  $\frac{x^2}{16} - \frac{y^2}{9} = 1$ .



## Asymptotes of a Hyperbola in Standard Position

To find equations for the asymptotes of the hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , it helps to examine the special case when  $a$  and  $b$  both equal 1. (This is like examining the unit circle to learn about ellipses.)

Then  $x^2 - y^2 = 1$ . The hyperbola with this equation is symmetric to both axes.

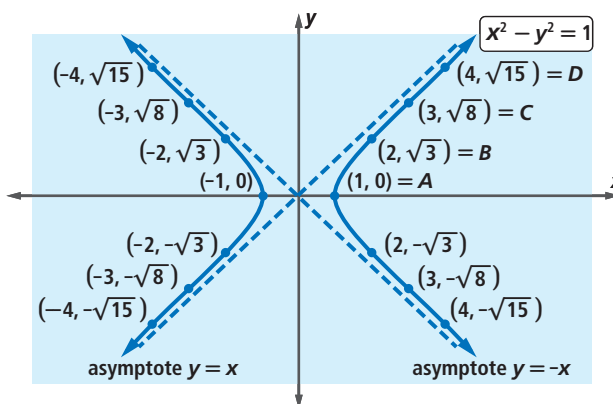
Consequently, each point on the hyperbola in the first quadrant has reflection images on the hyperbola in other quadrants. The graph at the right shows the reflection images of  $A$ ,  $B$ ,  $C$ , and  $D$  over the  $x$ -axis and the  $y$ -axis.

$$A = (1, 0)$$

$$B = (2, \sqrt{3}) \approx (2, 1.73)$$

$$C = (3, \sqrt{8}) \approx (3, 2.83)$$

$$D = (4, \sqrt{15}) \approx (4, 3.87)$$



The lines  $y = -x$  and  $y = x$  appear to be the asymptotes of  $x^2 - y^2 = 1$ . We can verify the equations for the asymptotes algebraically.

When  $x^2 - y^2 = 1$ ,

$$y^2 = x^2 - 1.$$

So  $y = \pm \sqrt{x^2 - 1}$ .

As values of  $x$  get larger,  $\sqrt{x^2 - 1}$  becomes closer to  $\sqrt{x^2}$ , which is  $|x|$ . However, because  $\sqrt{x^2 - 1} \neq \sqrt{x^2}$ , the curve  $x^2 - y^2 = 1$  never intersects the lines with equations  $y = x$  or  $y = -x$ . So,  $y$  gets closer to  $x$  or  $-x$  but never reaches it.

According to the Graph Scale-Change Theorem, the scale change

$S_{a,b}$  maps  $x^2 - y^2 = 1$  onto  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . Under the same scale

change, the asymptotes  $y = \pm x$  of  $x^2 - y^2 = 1$  are mapped onto the lines with equations  $\frac{y}{b} = \pm \frac{x}{a}$ . These lines are the asymptotes of  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .

### Asymptotes of a Hyperbola Theorem

The asymptotes of the hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are  $\frac{y}{b} = \pm \frac{x}{a}$ , or  $y = \pm \frac{b}{a}x$ .

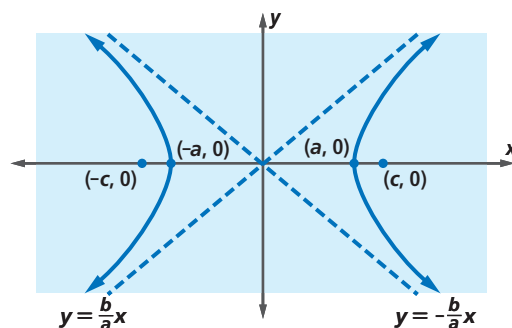
#### STOP QY

#### QY

What are the asymptotes of the hyperbola in Example 1?

## Graphing a Hyperbola with Equation in Standard Form

To graph  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  by hand, notice that  $(a, 0)$  and  $(-a, 0)$  satisfy the equation. These are the vertices of the hyperbola. When  $x = 0$ ,  $y$  is not a real number, so the hyperbola does not intersect the  $y$ -axis. Use the asymptotes to make an accurate sketch of the graph. Remember that the asymptotes are not part of the hyperbola.



### Example 2

Graph the hyperbola with equation  $\frac{x^2}{16} - \frac{y^2}{36} = 1$ .

**Solution** The equation is in standard form. So,  $a^2 = 16$  and  $a = 4$ . The vertices are  $(4, 0)$  and  $(-4, 0)$ . The asymptotes are  $\frac{y}{6} = \pm \frac{x}{4}$ , or  $y = \pm \frac{3}{2}x$ . Carefully graph the vertices and asymptotes. Then sketch the hyperbola.

**Check** Solve  $\frac{x^2}{16} - \frac{y^2}{36} = 1$  for  $y$  on a CAS.

One CAS solution is shown below.

$$\text{solve}\left(\frac{x^2}{16} - \frac{y^2}{36} = 1, y\right)$$

$$y = \frac{3 \cdot \sqrt{x^2 - 16}}{2} \text{ and } x^2 - 16 \geq 0 \text{ or } y = -\frac{3 \cdot \sqrt{x^2 - 16}}{2}$$

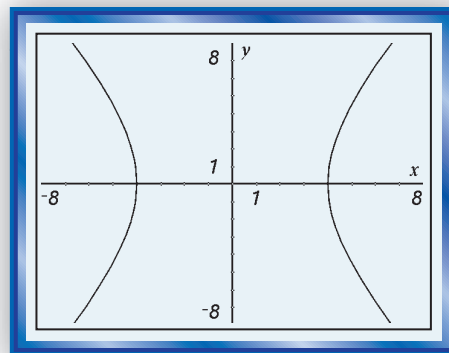
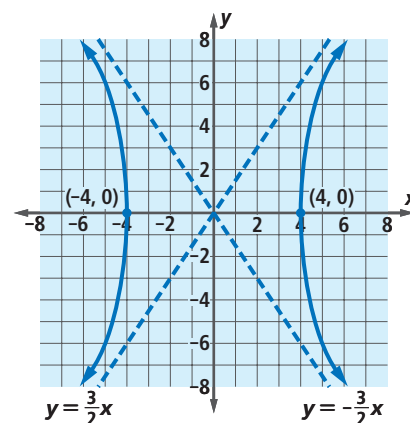
The complete solution is

$$y = \frac{3 \cdot \sqrt{x^2 - 16}}{2} \text{ and } x^2 - 16 \geq 0 \text{ or } y = -\frac{3 \cdot \sqrt{x^2 - 16}}{2} \text{ and } x^2 - 16 \geq 0.$$

$$\text{So } y = \frac{3\sqrt{x^2 - 16}}{2} \text{ or } y = -\frac{3\sqrt{x^2 - 16}}{2}.$$

Graph both equations on the same axes on a graphing utility.

Although the graphing utility may have trouble graphing values close to the vertices of the hyperbola, the output closely resembles the hand-drawn solution.



## Questions

### COVERING THE IDEAS

- Fill in the Blanks** A hyperbola with foci  $(c, 0)$  and  $(-c, 0)$  and focal constant  $2a$  has an equation of the form  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and vertices at  $(\pm a, 0)$  and  $(0, \pm b)$ .
- Fill in the Blanks** A hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  has asymptotes  $y = \pm \frac{b}{a}x$  and  $y = \pm \frac{a}{b}x$ .

In 3 and 4, an equation for a hyperbola is given. Identify its vertices, its foci, and its asymptotes.

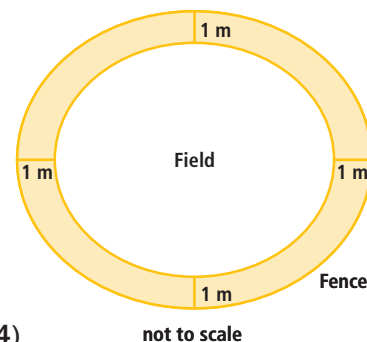
- $1 = x^2 - y^2$
- $\frac{x^2}{7^2} - \frac{y^2}{3^2} = 1$
- True or False** The focal constant of a hyperbola equals the distance between the foci.
- True or False** If  $F_1$  and  $F_2$  are the foci of a hyperbola, then  $\overleftrightarrow{F_1F_2}$  is a line of symmetry for the curve.
- What does the phrase “ $\sqrt{x^2 - 1}$  is close to  $\sqrt{x^2}$  for large values of  $x$ ” mean?
- Consider the hyperbola with equation  $\frac{x^2}{25} - \frac{y^2}{64} = 1$ .
  - Name its vertices and state equations for its asymptotes.
  - Graph the hyperbola.

### APPLYING THE MATHEMATICS

- Explain why  $y = |x|$  is not an equation describing the asymptotes of  $x^2 - y^2 = 1$ .
- Write an equation for the hyperbola with vertices at  $(4, 0)$  and  $(-4, 0)$  and one focus at  $(7, 0)$ .
- The point  $(-6, 3)$  is on a hyperbola with foci  $(4, 0)$  and  $(-4, 0)$ .
  - Find the focal constant of the hyperbola.
  - Give an equation for this hyperbola in standard form. (*Hint: Find  $b$  using  $b^2 = c^2 - a^2$ .)*
  - Graph this hyperbola.
- Show that  $\frac{x^2}{91} - \frac{y^2}{49} = 1$  is equivalent to an equation of the general form  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  by finding the values of  $A, B, C, D, E,$  and  $F$ .
- Solve  $x^2 - y^2 = 1$  for  $y$ . Use your solution to graph  $x^2 - y^2 = 1$  on a graphing utility.

## REVIEW

14. In Australia, a type of football is played on elliptical fields. One such field has a major axis of length 185 meters and minor axis of length 155 meters. Surrounding it is an elliptical fence with major axis of length 187 meters and minor axis of length 157 meters. The 1-meter wide track between the fence and the field is to be covered with turf. Find the area of the track. (Lesson 12-5)



In 15 and 16, graph the ellipse with the given equation. (Lesson 12-4)

15.  $\frac{x^2}{4} + \frac{y^2}{25} = 1$

16.  $\frac{x^2}{9} + y^2 = 1$

17. Standard Quonset huts are semicircular with a diameter of 20 feet and a length of 48 feet. (Lesson 12-2)
- Inside the hut, how close to either side of the hut could a 6-foot soldier stand upright?
  - What is the volume of a hut?



During World War II, easy-to-build Quonset huts were used as barracks for troops.

18. An auto dealer is having a Fourth of July extravaganza. The dealership plans to be open for 72 hours straight. Suppose the dealer has 100 new cars on the lot and is able to sell an average of 4 cars every 3 hours. (Lesson 3-1)

- Let  $h$  be the number of hours the car dealership has been open and let  $C$  be the number of cars remaining on the lot. Find three other pairs of values that satisfy this relation and complete the table.
- Write a formula for the number of cars  $C$  on the lot as a function of the number of hours  $h$  the sale has been on.
- After how many hours will there be only 60 cars left?
- If the dealership is able to maintain the pace of 4 cars sold every 3 hours, will the dealer sell all the cars on the lot during the sale? How can you tell?

$h$	0	3	?	?	?
$C$	100	96	?	?	?

## EXPLORATION

19. The words *ellipsis* and *hyperbole* have literary meanings. What are these meanings?
20. In *Round the Moon*, a novel written by Jules Verne in 1870, a group of men launch a rocket to the Moon. During the journey they argue whether the rocket trajectory is hyperbolic or parabolic. Because each curve is infinite, the men believe they are doomed to travel infinitely through space. Find out on which trajectory modern day rockets travel and whether or not the men had reason to worry.

## QY ANSWER

$$y = \pm \frac{3}{4}x$$