

Lesson

12-5

Relationships
between Ellipses
and Circles

Vocabulary

eccentricity of an ellipse

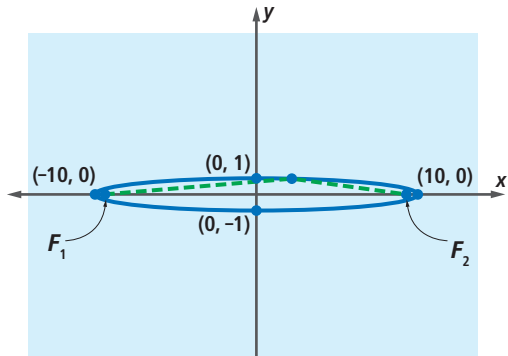
► **BIG IDEA** Ellipses are stretched circles; circles are ellipses whose major and minor axes have the same length.

In some ellipses, the major axis is much longer than the minor axis. In others, the two axes are almost equal in length. The diagrams below illustrate three of the possible cases. All three ellipses have the same focal constant. You can see that when the focal constant stays the same, the positions of the foci in an ellipse affect its shape. The closer the foci are to the origin, the rounder the ellipse appears.

Mental Math

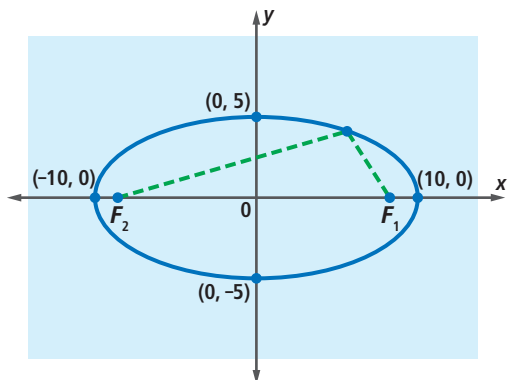
Describe a transformation or composite of transformations that will map (x, y) onto the given point.

- $(x + 7, y - 2)$
- $(3x, 2.5y)$
- $(3x + 7, 2.5y - 2)$
- $(2.5y - 2, 3x + 7)$



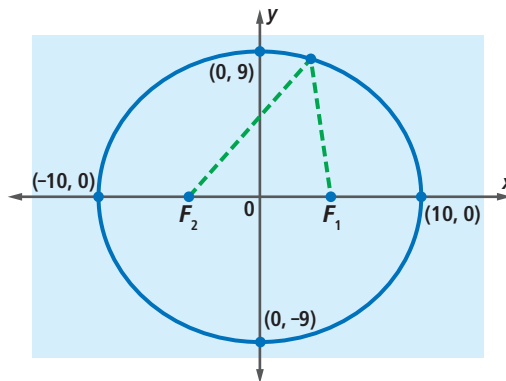
$$\frac{x^2}{100} + \frac{y^2}{1} = 1$$

Major Axis: 20
Minor Axis: 2



$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

Major Axis: 20
Minor Axis: 10



$$\frac{x^2}{100} + \frac{y^2}{81} = 1$$

Major Axis: 20
Minor Axis: 18

Circles as Special Ellipses

A special kind of ellipse results if the major and minor axes are equal. At the right is an ellipse with major axis 20 and minor axis 20. It has equation

$$\frac{x^2}{100} + \frac{y^2}{100} = 1,$$

which can be rewritten as

$$x^2 + y^2 = 100.$$

So this ellipse is a circle.

This can be generalized. Consider the standard form of an equation for an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If the major and minor axes each have length $2r$, then $2a = 2r$ and $2b = 2r$, so you may substitute r for both a and b to get $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$.

Multiply both sides by r^2 to get $x^2 + y^2 = r^2$.

This is an equation for the circle with center at the origin and radius r . So, a circle is a special kind of ellipse whose major and minor axes are equal in length.

STOP QY1

Ellipses as Stretched Circles

An ellipse can be thought of as a stretched circle. The transformation that stretches and shrinks figures is the *scale change*, which you studied in Lesson 4-5.

Consider the unit circle with equation $x^2 + y^2 = 1$ and a scale change with horizontal magnitude 4 and vertical magnitude 2. The x - and y -intercepts of the unit circle and their images under this scale change are graphed at the right.

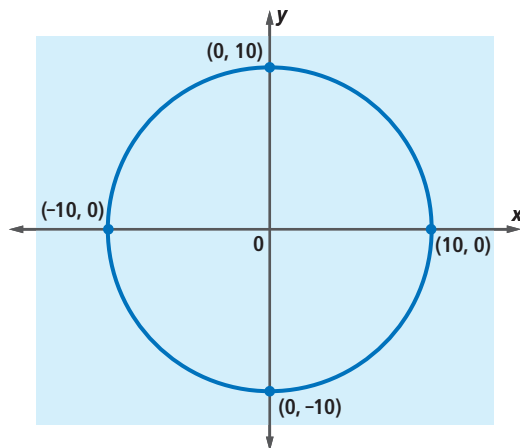
$$S_{4,2}: (1, 0) \rightarrow (4, 0)$$

$$S_{4,2}: (0, 1) \rightarrow (0, 2)$$

$$S_{4,2}: (-1, 0) \rightarrow (-4, 0)$$

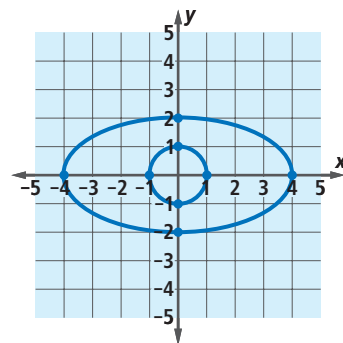
$$S_{4,2}: (0, -1) \rightarrow (0, -2)$$

From these four points you can see that the image of the unit circle under this scale change is not a circle. It appears to be an ellipse with foci on the x -axis.



► QY1

Find an equation of the ellipse with center at the origin and major and minor axes of length 14.



GUIDED

Example

Find an equation for the image of the unit circle $x^2 + y^2 = 1$ under $S_{4,2}$.

Solution To find an equation for the image of the circle, let (x', y') be the image of point (x, y) on the circle. Then $(x', y') = (4x, 2y)$. So, $x' = 4x$ and $y' = \underline{\quad?}$.

Solve these equations for x and y .

$$x = \underline{\quad?} \text{ and } y = \underline{\quad?}$$

You know that $x^2 + y^2 = 1$. Substitute the expressions for x and y involving x' and y' into $x^2 + y^2 = 1$ to get an equation for the image.

$$(\underline{\quad?})^2 + (\underline{\quad?})^2 = 1.$$

Now let (x, y) be a point on the image. Rewrite the equation for the image using x and y in place of x' and y' .

$$(\underline{\quad?})^2 + (\underline{\quad?})^2 = 1$$

The equation you have written is for an ellipse with a minor axis of length $\underline{\quad?}$ and a major axis of length $\underline{\quad?}$.

Check Substitute some points known to be on the image into the equation. Do their coordinates satisfy your equation for the ellipse?

Try $(4, 0)$: $\frac{4^2}{\underline{\quad?}} + \frac{0^2}{\underline{\quad?}} = \underline{\quad?} + \underline{\quad?} = \underline{\quad?}$.

Try $(0, -2)$: $\frac{0^2}{\underline{\quad?}} + \frac{(-2)^2}{\underline{\quad?}} = \underline{\quad?} + \underline{\quad?} = \underline{\quad?}$. It checks.

The procedure you followed in the Example can be repeated using a in place of 4 and b in place of 2. This shows that any ellipse in standard form can be thought of as a scale-change image of the unit circle.

Circle Scale-Change Theorem

The image of the unit circle with equation $x^2 + y^2 = 1$ under $S_{a,b}$ is the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

The previous theorem is a special case of a more general Graph Scale-Change Theorem, which is analogous to the Graph-Translation Theorem you studied in Chapter 6.

Graph Scale-Change Theorem

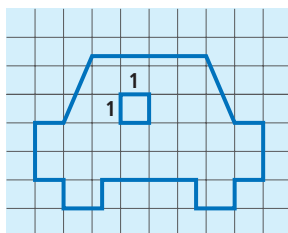
In a relation described by a sentence in x and y , the following two processes yield the same graph:

1. replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$;
2. applying the scale change $S_{a,b}$ to the graph of the original relation.

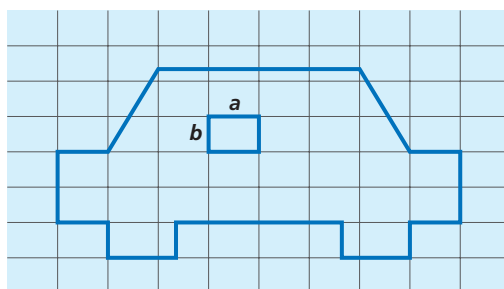
STOP QY2

A Formula for the Area of an Ellipse

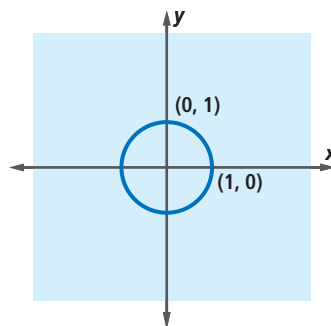
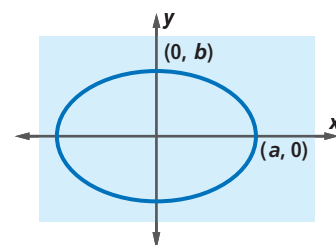
Consider the figure below at the left. If each grid square has area 1, the area of the figure is equal to the number of grid squares inside the figure. Now suppose the scale change $S_{a,b}$ is applied to the figure and the grid. The result is the figure at the right below. Each grid square is transformed into a rectangle with length a , width b , and area ab . Since the area of each rectangle is ab times the area of one grid square, the area of the transformed figure is ab times the area of its preimage.



preimage

image under $S_{a,b}$

This illustrates that in general, the area of the image of a figure under the scale change $S_{a,b}$ is ab times the area of the preimage. This fact can be used to derive a formula for the area of an ellipse. The area of the unit circle is $\pi(1)^2 = \pi$, and the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the image of the unit circle under $S_{a,b}$. So, the area of this ellipse is $\pi(ab) = \pi ab$.

area: π square unitsarea: πab square units

QY2

Find an equation for the image of $x^2 + y^2 = 1$ under $S_{6,14}$.

Area of an Ellipse Theorem

An ellipse with semimajor and semiminor axes of lengths a and b has area $A = \pi ab$.

STOP QY3

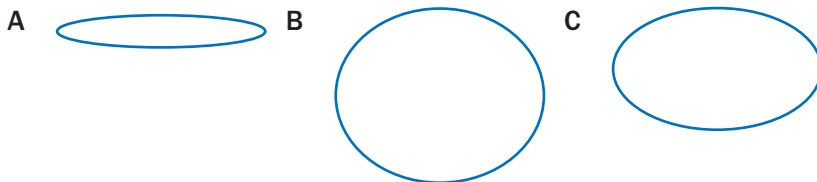
QY3

The Statuary Hall gallery in the United States Capitol is an elliptical chamber. The gallery is about 83 feet wide and about 96 feet long. Find its floor area.

Questions

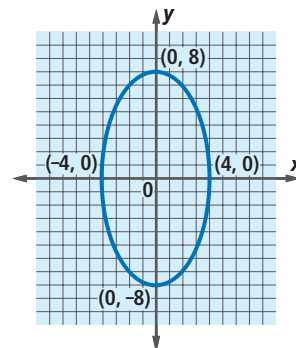
COVERING THE IDEAS

- Fill in the Blank** A circle is an ellipse in which the major and minor axes
 - are parallel.
 - are perpendicular.
 - are of equal length.
 - coincide.
- True or False** In 2 and 3, indicate whether the statement is true or false. If false, draw a counterexample.
 - All ellipses are circles.
 - All circles are ellipses.
- Fill in the Blank** All three ellipses below are the same width. In which ellipse are the foci farthest apart? Explain your answer.



In 5 and 6, consider the circle $x^2 + y^2 = 1$ and the scale change $S_{5, \frac{1}{2}}$.

- Find the image of $(1, 0)$ under $S_{5, \frac{1}{2}}$.
 - Find the image of $(0, 1)$ under $S_{5, \frac{1}{2}}$.
 - Write an equation of the image of the circle under $S_{5, \frac{1}{2}}$.
- What is the area of the image?
- Consider the ellipse drawn at the right.
 - What scale change maps the unit circle onto this ellipse?
 - Write an equation for the ellipse.
 - Find its area.



APPLYING THE MATHEMATICS

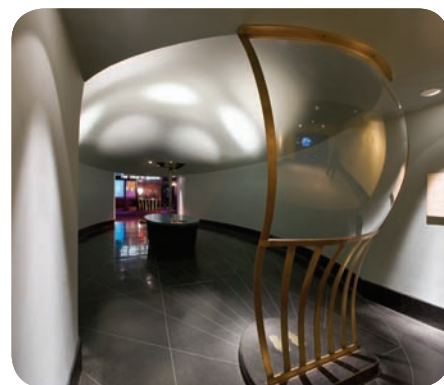
8. a. Write an equation for an ellipse in which the semimajor and semiminor axes each have length 5 and the distance between the foci is 0.
b. What relationship between ellipses and circles does this illustrate?
9. a. Give an example of a scale change $S_{h,k}$ such that the unit circle and its image under the scale change are similar.
b. Give an example of a scale change $S_{h,k}$ such that the unit circle and its image under the scale change are not similar.
10. Use the method of the Example to prove that the image of the unit circle under the scale change $S_{a,b}$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
11. The orbit of the planet Mars is in the shape of an ellipse whose minor axis is 282.0 million miles long and whose major axis is 283.3 million miles long. What is the area of this ellipse?

In 12–14, use this definition: The **eccentricity** of an ellipse is the ratio of the distance $2c$ between its foci to the length $2a$ of its major axis.

12. a. Write a formula for the eccentricity e of an ellipse.
b. What is the eccentricity of the ellipse in the Example?
13. An ellipse's major axis is 12 units long. If its eccentricity is $\frac{1}{3}$, how long is its minor axis?
14. a. What is the eccentricity of a circle?
b. Is there a maximum possible value for the eccentricity of an ellipse? Why or why not?
15. Write the equations of four different ellipses with area 24π .

REVIEW

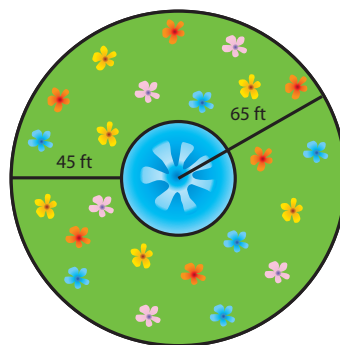
16. In 1937, a Whispering Gallery was constructed in the Museum of Science and Industry in Chicago. The Gallery was constructed in the form of an *ellipsoid* (an ellipse rotated around its major axis). When a visitor located at one focus whispers, the sound reflects directly to the focus at the other end of the gallery. The width of an ellipse in the plane of the foci is 13 feet 6 inches, and the length of the ellipse is 47 feet 4 inches.
 - a. Find an equation that could describe this ellipse.
 - b. How far are the foci from the endpoints of the major axis of this ellipse? (Lesson 12-4)



Matching In 17–21, match each equation with the best description. A letter may be used more than once. Do not graph. (Lessons 12-4, 12-3, 12-2)

- | | |
|---|---------------------------|
| 17. $(x - 2)^2 + y^2 < 36$ | i circle |
| 18. $\frac{x^2}{48} + \frac{y^2}{64} > 1$ | ii ellipse |
| 19. $x^2 + 9y^2 = 121$ | iii interior of a circle |
| 20. $\frac{x^2}{16} + \frac{y^2}{81} = 4$ | iv interior of an ellipse |
| 21. $x^2 + y^2 = 25$ | v exterior of a circle |
| | vi exterior of an ellipse |

22. At the right is a top view of a circular fountain surrounded by a circular flower garden. The distance from the center of the fountain to the outside edge of the garden is 65 feet. It is 45 feet from the outside edge of the garden to the fountain. If the center of the fountain is the origin, write a system of inequalities to describe the set of points in the flower garden. (Lesson 12-3)



In 23 and 24, solve. (Lesson 6-2)

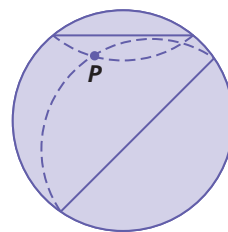
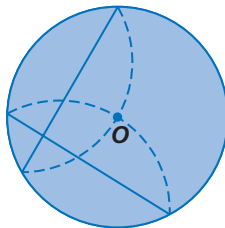
23. $|2n + 1| = 0.5$ 24. $\sqrt{x^2} = \sqrt{40}$

In 25 and 26, consider the line ℓ with equation $y = -\frac{3}{7}x + 6$ and the point $P = (5, 2)$. Find an equation for the line through P

25. parallel to ℓ . (Lesson 4-10)
26. perpendicular to ℓ . (Lesson 4-9)

EXPLORATION

27. a. Cut out a paper circle and make a dot anywhere in its interior. Fold a point on the circle onto the dot then unfold. Repeat with other points on the circle. What shape is outlined by the crease lines (shown as segments in the diagram)?
- b. Use a new circle. Place the dot at the center of the circle and repeat the activity in Part a.
- c. Explain your results in Parts a and b.



QY ANSWERS

- $\frac{x^2}{49} + \frac{y^2}{49} = 1$ or $x^2 + y^2 = 49$
- $\frac{x^2}{36} + \frac{y^2}{196} = 1$
- $1992\pi \approx 6258 \text{ ft}^2$