Lesson 12-4 Ellipses

Vocabulary

ellipse
foci, focal constant of an ellipse
standard position for an ellipse
standard form of an equation for an ellipse
major axis, minor axis
center of an ellipse
semimajor axes, semiminor axes

BIG IDEA From the geometric definition of an ellipse, an equation for any ellipse symmetric to the *x*- and *y*-axes can be found.

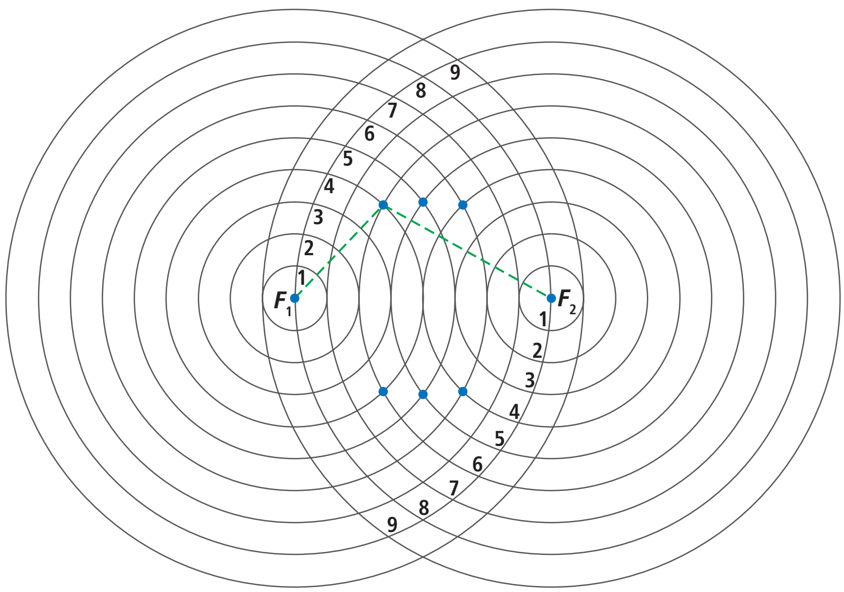
On the first page of the chapter, we noted that when a plane intersects a cone but is not parallel to the cone's edge, the intersection is a figure called an *ellipse*. Drawing an ellipse freehand is challenging. Identifying several points on the ellipse and connecting them with a smooth curve simplifies the task. The following Activity provides one method for sketching an ellipse.

Activity

MATERIALS conic graph paper with 8 units between the centers of the circles

Step 1 Label the centers of the circles as F_1 and F_2. Plot the two points of intersection of the circles that are 5 units from both F_1 and F_2.

Step 2 a. Mark the two points of intersection between circles that are 4 units from F_1 and 6 units from F_2.
b. Mark the two points of intersection between circles that are 6 units from F_1 and 4 units from F_2.



Mental Math

At the state championship basketball game, the Beatums are beating the Underdogs. The score is 84-52. If the Beatums do not score any more points,

a. how many 2-point shots must the Underdogs make to win?

b. how many 3-point shots must they make to win?

c. how many 2-point shots must they make to win if they also make five 3-point shots?

Step 3 Repeat the process of Step 2 plotting the intersections of circles that are x units from F_1 and $(10 - x)$ units from F_2 until you have 16 points plotted.

Step 4 Starting on the left and moving clockwise, label the points $P_1, P_2, \ldots, P_{16}$. Connect the dots with a smooth curve to form an ellipse.

What Is an Ellipse?

The ellipse in the Activity is determined by the two points F_1 and F_2, called its *foci* (pronounced "foe sigh," plural of focus), and a number called the *focal constant*. The focal constant is the constant sum of the distances from any point P on the ellipse to the foci. The focal constant of the ellipse in the Activity is 10. That is, $P_nF_1 + P_nF_2 = 10$ for all points P_n on the curve. Every ellipse can be determined in this way.

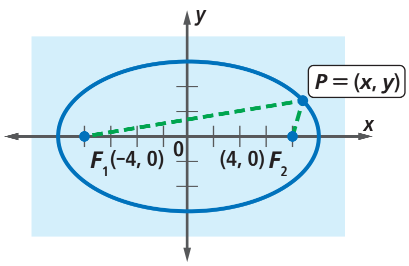
Definition of Ellipse

Let F_1 and F_2 be any two points in a plane and let d be a constant with $d > F_1F_2$. Then the **ellipse** with **foci** F_1 and F_2 and **focal constant** d is the set of points P in the plane for which $PF_1 + PF_2 = d$.

The equation $PF_1 + PF_2 = d$ is said to define the ellipse. For any point P on the ellipse, the focal constant $PF_1 + PF_2$ has to be greater than F_1F_2 because of the Triangle Inequality. That is why $d > F_1F_2$.

Equations for Some Ellipses

To find an equation for the ellipse in the Activity, consider a coordinate system with $\overleftrightarrow{F_1F_2}$ as the x-axis and with the origin midway between the foci on the axis. Then the foci are $F_1 = (-4, 0)$ and $F_2 = (4, 0)$. This is the *standard position* for the ellipse.



If $P = (x, y)$ is on the ellipse, then because the focal constant is 10,

$$PF_1 + PF_2 = 10.$$

So, by the Distance Formula,

$$\sqrt{(x+4)^2 + (y-0)^2} + \sqrt{(x-4)^2 + (y-0)^2} = 10,$$

or

$$\sqrt{(x+4)^2 + y^2} + \sqrt{(x-4)^2 + y^2} = 10.$$

This equation for the ellipse is quite involved. Surprisingly, to find an equation for this ellipse, and all others with their foci on an axis, it is simpler to begin with a more general case. The resulting equation is well worth the effort it takes to derive it, and so on the next page we state it as a theorem. You are asked to justify the numbered steps in the proof of this theorem in Question 14.

Equation for an Ellipse Theorem

The ellipse with foci $(c, 0)$ and $(-c, 0)$ and focal constant $2a$ has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$.

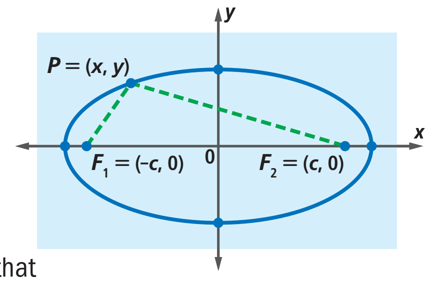
Proof Let $F_1 = (-c, 0)$, $F_2 = (c, 0)$, and $P = (x, y)$. By the definition of an ellipse,

$$PF_1 + PF_2 = 2a.$$

1. $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$
2. $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$
3. $(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$
4. $-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx$
5. $4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$
6. $a\sqrt{(x+c)^2 + y^2} = a^2 + cx$
7. $a^2((x+c)^2 + y^2) = a^4 + 2a^2cx + c^2x^2$
8. $a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2$
9. $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$
10. Because $c > 0$, $F_1F_2 = 2c$, and $2a > F_1F_2$, you can conclude that $2a > 2c > 0$, so $a > c > 0$. Thus, $a^2 > c^2$, and $a^2 - c^2$ is not negative. So $a^2 - c^2$ can be considered as the square of some real number, say b. Let $b^2 = a^2 - c^2$ and substitute to get

$$b^2x^2 + a^2y^2 = a^2b^2.$$

11. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The Standard Form of an Equation for an Ellipse in Standard Position

An ellipse centered at the origin with its foci on an axis is in **standard position**, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is in the **standard form** of an equation for this ellipse. Even without knowing the geometric interpretation of this equation, analyzing the formula can tell you a lot about the shape of its graph.

GUIDED

Example 1

Given the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, describe

a. the x- and y-intercepts of its graph.

b. the possible x- and y-values.

c. the symmetries of the graph, if any.

Solution

a. The x-intercepts are the x-values for which $y = 0$. Substitute 0 for y to get $\frac{x^2}{9} + \frac{0^2}{4} = 1$. Then solve for x to get $x =$ __?__. Similarly, the y-intercepts are the y-values for which $x =$ __?__. Substitute and solve for y to get $y =$ __?__.

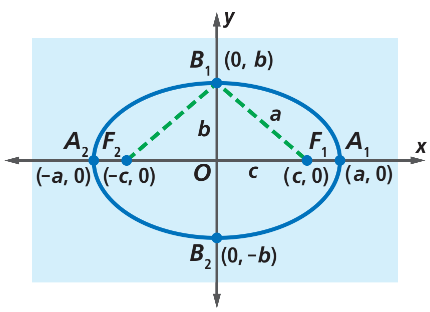
b. Now $y^2 \geq 0$ and $\frac{x^2}{9} \leq 1$, so $x^2 \leq 9$. Therefore, $\sqrt{x^2} \leq 3$ and $|x| \leq 3$, so $-3 \leq x \leq 3$. Similarly, $x^2 \geq 0$ and $\frac{y^2}{4} \leq$ __?__, so __?__ $\leq y \leq$ __?__.

c. Recall that r_y: $(x, y) \rightarrow (-x, y)$. If you replace x with $-x$ in the equation, it becomes $\frac{(-x)^2}{9} + \frac{y^2}{4} = 1$. Because $x^2 = (-x)^2$, this equation is equivalent to the original. Therefore, the graph is symmetric to the __?__.

To check for symmetry over the x-axis, recall that r_x: $(x, y) \rightarrow$ __?__. So, if you replace __?__ with __?__ in the original equation, it becomes __?__, which is equivalent to the original. Therefore, the graph is symmetric to the __?__ as well.

Generalizing from Example 1, the intercepts of the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $(a, 0)$, $(-a, 0)$, $(0, b)$ and $(0, -b)$. The possible x- and y-values indicate that the entire ellipse is contained in the rectangle $\{(x, y): -a \leq x \leq a \text{ and } -b \leq y \leq b\}$. These facts can help you sketch a graph of an ellipse by hand.

Consider the ellipse at the right. The segments $\overline{A_1A_2}$ and $\overline{B_1B_2}$ are, respectively, the **major** and **minor axes** of the ellipse. The major axis contains the foci and is always longer than the minor axis. The axes lie on the symmetry lines and intersect at the **center** O of the ellipse. Each segment $\overline{OA_1}$ and $\overline{OA_2}$ is a **semimajor axis** of the ellipse. The segments $\overline{OB_1}$ and $\overline{OB_2}$ are the **semiminor axes** of the ellipse. The diagram illustrates the following theorem. It applies to all ellipses centered at the origin with foci on one of the coordinate axes.



Length of Axes of an Ellipse Theorem

In the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $2a$ is the length of the horizontal axis, and $2b$ is the length of the vertical axis.

The length of the major axis is the focal constant. If $a > b$, then the major axis is horizontal and $(c, 0)$ and $(-c, 0)$ are the foci. The focal constant is $2a$, the length of the semimajor axis is a, and by the Pythagorean Theorem $b^2 = a^2 - c^2$ (as in the ellipse on the previous page).

If $b > a$, then the major axis is vertical. So the foci are $(0, c)$ and $(0, -c)$, the focal constant is $2b$, and $a^2 = b^2 - c^2$.

STOP QY

QY

In the ellipse of Example 1, what are the lengths of the major and minor axes?

GUIDED

Example 2

Find the endpoints of the major and minor axes, the foci, and an equation for the ellipse in the Activity in standard position.

Solution Position a coordinate system on the ellipse, with origin at the center of the ellipse and the foci on the x-axis. Let the radius of the smallest circle on the conic graph paper be 1 unit on the axes scales.

The major axis has length __?__, so a = __?__.

The minor axis has length __?__, so b = __?__.

Therefore, in standard position, the endpoints of the major axis of the ellipse are (__?__, 0) and (__?__, 0), and the endpoints of the minor axis are (0, __?__) and (0, __?__).

Since $a > b$, $b^2 = a^2 - c^2$. So $9 = 25 - c^2$ and $c = 4$.

The foci for the ellipse are (__?__, 0) and (__?__, 0).

An equation for this ellipse is $\frac{x^2}{?} + \frac{y^2}{?} = 1$.

Graphing an Ellipse in Standard Form

Example 3

Consider the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{10} = 1$.

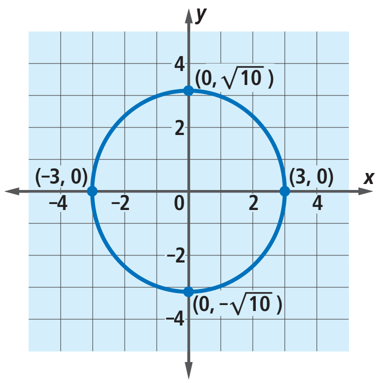
a. Identify the endpoints and the lengths of the major and minor axes.

b. Graph the ellipse.

Solution

a. $a^2 = 9$ and $b^2 = 10$. So $a = 3$ and $b = \sqrt{10}$. Because $b > a$, the foci of the ellipse are on the y-axis. The endpoints of the major axis are $(0, \sqrt{10})$, and $(0, -\sqrt{10})$. The endpoints of the minor axis are $(3, 0)$, and $(-3, 0)$. The length of the major axis is $2\sqrt{10}$ and the length of the minor axis is 6.

b. Plot the four axis endpoints, then sketch the rest of the ellipse. A graph of the ellipse is at the right.



Questions

COVERING THE IDEAS

In 1 and 2, refer to the ellipse in the Activity.

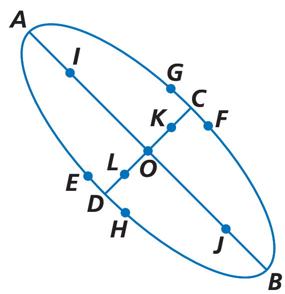
1. What is the focal constant?

2. What is the length of each

   a. semimajor axis? b. semiminor axis?

3. On the ellipse at the right, $OA = OB$, $OD = OC$ and $\overline{AB} \perp \overline{CD}$. Identify its

   a. foci. b. major axis. c. minor axis.



In 4–8, consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Assume $b > a$. Identify the following.

4. the center of the ellipse

5. the endpoints of the major and minor axes

6. the lengths of the semimajor and semiminor axes

7. the possible values of x and the possible values of y

8. the x- and y-intercepts

9. Sketch a graph of the ellipse with equation $\frac{x}{16} + y^2 = 1$.

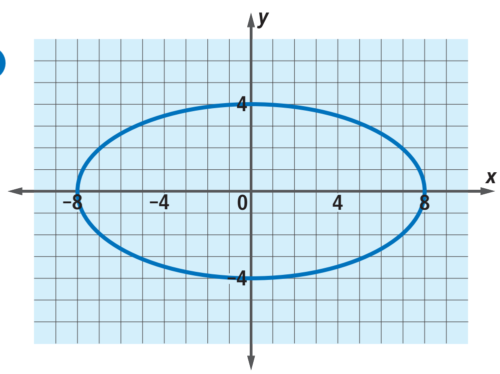
10. Write an equation in standard form for the ellipse with focal constant 17 and foci $(8, 0)$ and $(-8, 0)$.

APPLYING THE MATHEMATICS

11. Refer to the ellipse graphed at the right.

    a. Find an equation for the ellipse.

    b. Write an inequality describing the points in the interior of the ellipse.

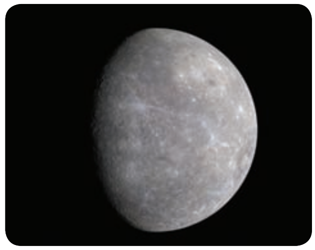


12. a. Write an equation describing the set of points (x, y) whose distances from $(-1, -1)$ and $(1, 1)$ add up to 4.
    b. Use a CAS to solve your equation in Part a for y.
    c. Graph the equation on a graphing utility and sketch the result.

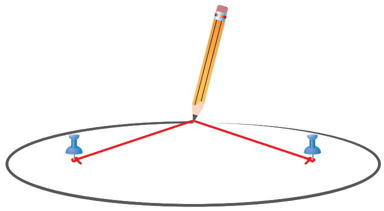
13. a. Find the foci and focal constant of the ellipse with equation $\frac{x^2}{100} + \frac{y^2}{100} = 1$.
    b. What is special about this ellipse?

14. Write justifications for the statements in the proof of the Equation for an Ellipse Theorem.

15. The orbits of the planets are nearly elliptical with the Sun at one focus. The shape of Mercury's orbit can be approximated by the equation $\frac{x^2}{1295} + \frac{y^2}{1240} = 1$, where x and y are in millions of miles.
    a. What is the farthest Mercury gets from the Sun?
    b. What is the closest Mercury gets to the Sun?
    c. How far apart are the two foci?



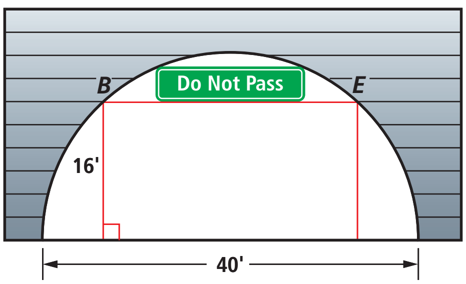
16. a. Using two thumbtacks and a piece of string, draw a curve as shown at the right.
    b. Explain why the curve is an ellipse.
    c. What part of your equipment represents the focal constant of the ellipse?



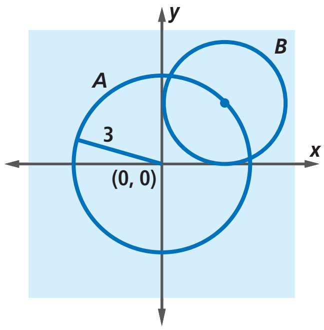
17. A *superellipse* has equation $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$, for nonzero a and b and $n > 0$.
    a. Sketch a graph of the superellipse for $a = 2$, $b = 3$, and $n = 1$.
    b. Identify the major and minor axis on your sketch in Part a.
    c. List the x- and y- intercepts of the graph.
    d. Repeat Part a when $a = 2$, $b = 3$, and $n = 3$.
    e. How are your graphs in Parts a and d related? How are they different?

REVIEW

18. The figure at the right shows a cross section of a tunnel with diameter 40 feet. A rectangular sign reading "Do Not Pass" must be placed at least 16 feet above the roadway. **(Lesson 12-3)**
    a. Find the length BE of the beam that supports the sign.
    b. If the sign is 16 feet long, what is its maximum height?

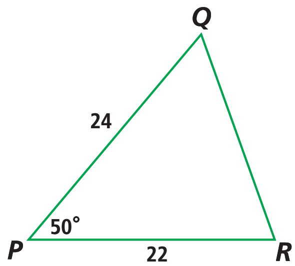


19. Refer to the diagram at the right. Circle A is centered at the origin with radius 3. Circle B is tangent to both axes and has its center on circle A. **(Lessons 12-3, 12-2, Previous Course)**



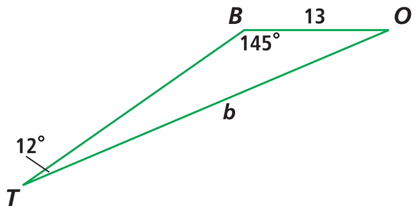
   a. Write an equation for circle A.
   b. Write an equation for circle B.
   c. Find an inequality describing the interior of circle A.
   d. Find the circumference of each circle.
   e. Find the area of each circle.

20. Use $\triangle PQR$ at the right. Find the length of $\overline{QR}$ to the nearest tenth. **(Lesson 10-8)**

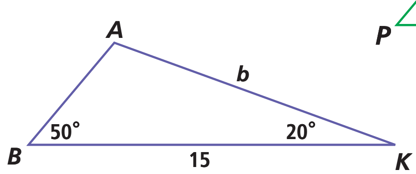


In 21 and 22, find *b*. (Lesson 10-7)

21.



22.



23. What is the image of (x, y) under the scale change $S_{7,8}$? **(Lesson 4-5)**

EXPLORATION

24. Over the years, many devices were invented to help drafters draw ellipses by hand. One such tool is the *carpenter's trammel*. Research the carpenter's trammel and explain how it is used to draw an ellipse.

QY ANSWER

major axis, 6; minor axis, 4