Lesson

12-3 Semicircles, Interiors, and Exteriors of Circles

Vocabulary

interior, exterior of a circle

BIG IDEA An inequality describing the points (x, y) in the *interior* or *exterior* of a circle can be found by replacing the equal sign in a circle's equation by $<$ or $>$.

Semicircles

In Lesson 12-2 you used two semicircles to create a graph of a circle on a graphing utility. In everyday life, semicircles often occur in design and architecture, as shown in the photo at the right of the Arc de Triomphe du Carrousel, located in Paris, France.

Mental Math

Suppose $P(x) = (x + 2)(x - 7)^2(2x + 3)$.

a. Give the degree of $P(x)$.

b. Determine all the roots of $P(x)$.

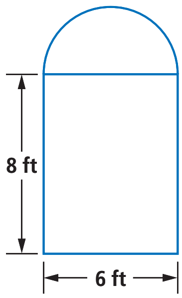
c. Determine all the roots with multiplicity greater than 1.

Example 1

An architect is designing a 6-foot wide and 8-foot tall passageway with a semicircular arch at its top.

a. Suppose the x-axis represents the floor with the origin at the center of the passageway. What is an equation for the semicircle representing the arch?

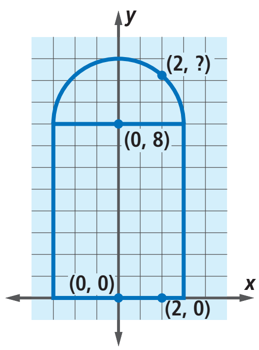
b. How high is the arch above a point on the ground 2 feet from the center of the passageway?



Solution

a. Let 1 unit in the coordinate system equal 1 foot. Then the circle whose top half is the arch has center (0, 8) and diameter 6, so its radius is 3. Therefore, its equation is $x^2 + (y - 8)^2 = 3^2$. Solve this equation for y.

$$x^2 + (y - 8)^2 = 9$$
$$(y - 8)^2 = 9 - x^2$$



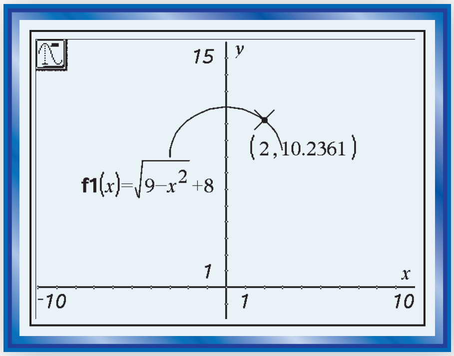
$$y - 8 = \pm\sqrt{9 - x^2}$$
$$y = \pm\sqrt{9 - x^2} + 8$$

Since we want the upper semicircle, $y = \sqrt{9 - x^2} + 8$ is the equation for the semicircle.

b. Since we have an equation for the semicircle, we can easily use it to find the height of the arch at any point. At a point 2 feet along the ground from the center of the passageway, $x = 2$ or $x = -2$. Substitute either x-value into the equation for the semicircle to find the corresponding y-value, which is the height of the arch above this point.

$$y = \sqrt{9 - 2^2} + 8$$
$$= \sqrt{5} + 8 \approx 10.24$$

The arch is about 10.24 feet high 2 feet from the center of the passageway.



Check Graph $y = \sqrt{9 - x^2} + 8$ using technology. Use the trace feature to estimate the y-value when $x = 2$. It checks.

In general, when the equation $(x - h)^2 + (y - k)^2 = r^2$ is solved for y, the result is a pair of equations in the form $y = \pm\sqrt{r^2 - (x - h)^2} + k$. The equation with the positive square root describes the upper semicircle, and the equation with the negative square root describes the lower semicircle.

STOP **QY**

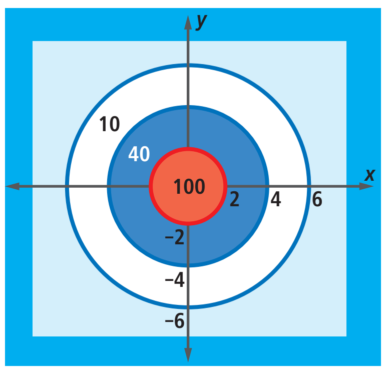
QY

Write an equation for the lower semicircle of the circle in Example 1.

Interiors and Exteriors of Circles

Every circle separates the plane into two regions. The region inside the circle is called the **interior of the circle**. The region outside the circle is called the **exterior of the circle**. The circle itself is the boundary between these two regions and is not part of either region.

Regions bounded by concentric circles are often used in target practice. Consider the target shown at the right.



To describe the colored regions mathematically, you can place the target on a coordinate system with the origin at the target's center. Notice that the region worth 100 points is the interior of a circle with radius 2. All points in this region are less than 2 units from the origin. Thus if (x, y) is a point in the 100-point region, then $\sqrt{x^2 + y^2} < 2$.

The expressions on both sides of this inequality are positive. Recall that whenever a and b are positive and $a < b$, then $a^2 < b^2$. Thus, when both sides of the inequality are squared, the sentence becomes $x^2 + y^2 < 4$, which also describes the 100-point region.

Similarly, the points in the region worth less than 100 points constitute the exterior of the circle with radius 2. All (x, y) in this region satisfy the sentence $\sqrt{x^2 + y^2} > 2$, or $x^2 + y^2 > 4$.

The two instances above are generalized in the following theorem.

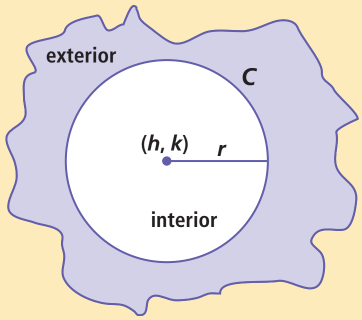
Interior and Exterior of a Circle Theorem

Let C be the circle with center (h, k) and radius r. Then the interior of C is described by

$$(x - h)^2 + (y - k)^2 < r^2$$

and the exterior of C is described by

$$(x - h)^2 + (y - k)^2 > r^2.$$



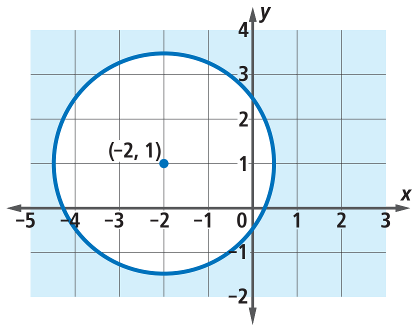
If $\geq$ or $\leq$ is used instead of $>$ or $<$ in the theorem above, the boundary (the circle itself) is included.

Example 2

Graph the points satisfying $(x + 2)^2 + (y - 1)^2 \geq 7$.

Solution The sentence represents the union of a circle, with center at (–2, 1) and radius $\sqrt{7}$, and its exterior. The shaded region and the circle at the right make up the graph.

Check Test a specific point. The point (0, 0) is not in the exterior, so it should not satisfy the inequality. This is the case, because $(0 + 2)^2 + (0 - 1)^2 = 5$ and $5 < 7$.



GUIDED

Example 3

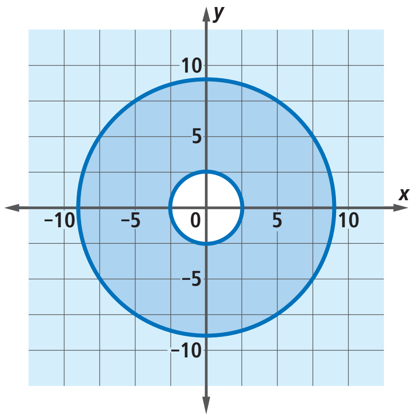
Some scholars argue that King Arthur's round table had a hole in it like the one shown at the right. An estimate of the diameter of the table is 18 feet. (The painting is not to scale.) Suppose the inner circle of the table has diameter 5 feet.

a. **Write a system of inequalities describing the tabletop.**

b. **Approximate the area of the tabletop to the nearest tenth of a square foot.**

Solution

a. Sketch a top view of the table, and superimpose a coordinate system with the center of the table at the origin.



The inner circle has center __?__, radius __?__ ft, and equation __?__. The outer circle has center __?__, radius __?__ ft, and equation __?__.

The tabletop is the intersection of the interior of the outer circle and the exterior of the inner circle, also including the circles themselves.

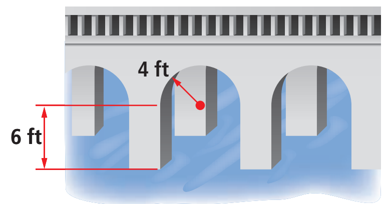
So, a system of inequalities describing the tabletop is $\begin{cases} \underline{\ ?\ } \geq \underline{\ ?\ } \\ \underline{\ ?\ } \leq \underline{\ ?\ } \end{cases}$.

b. The area of the tabletop is the difference of areas of the two circles. The area of the inner circle is __?__ ft^2. The area of the outer circle is __?__ ft^2. So, the area of the region bounded by these circles is __?__ − __?__ = __?__ ft^2 or about __?__ ft^2.

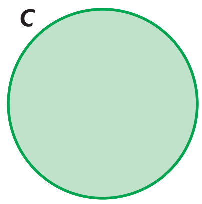
Questions

COVERING THE IDEAS

1. Sketch the graph of each equation by hand on separate axes.
   a. $x^2 + y^2 = 25$ b. $y = \sqrt{25 - x^2}$ c. $y = -\sqrt{25 - x^2}$
2. Graph $x^2 + (y - 4)^2 \leq 25$.
3. Use the equation for the arch found in Example 1. To the nearest inch, how high is the arch above a point on the ground 2 feet 6 inches from the center of the passage?
4. A bridge over water has a semicircular arch with radius 4 m set on pillars that extend 6 m above the water. How many meters above the water is the arch at a point 1.5 m from one of the pillars?

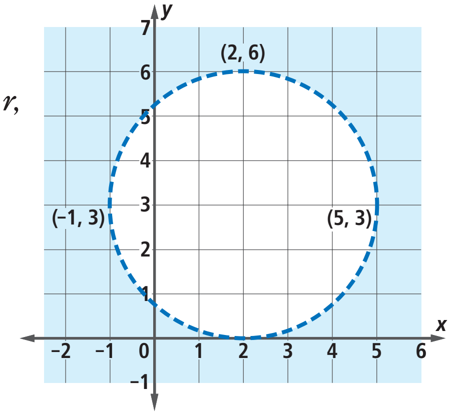


5. At the right, C is a circle. What is the shaded region called?



6. Refer to the target in this lesson. Write a system of inequalities to describe the set of points (x, y)
   a. in the 10-point region of the target.
   b. in the 40-point region of the target.

7. Write an inequality that describes all the points in the shaded region at the right.



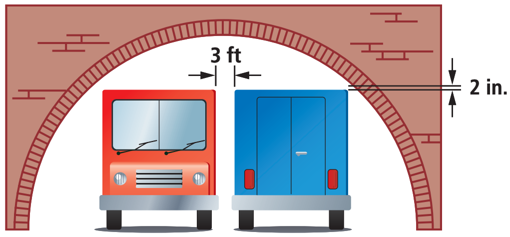
8. **Multiple Choice** Given a circle with center (h, k) and radius r, which of sentences A to D below describes
   a. the union of the circle with its interior?
   b. the exterior of the circle?

A $(x-h)^2 + (y-k)^2 > r^2$ B $(x-h)^2 + (y-k)^2 < r^2$

C $(x-h)^2 + (y-k)^2 \geq r^2$ D $(x-h)^2 + (y-k)^2 \leq r^2$

APPLYING THE MATHEMATICS

9. An architect designed a semicircular tunnel so that two trucks can pass each other with a 3-foot clearance between them and a 2-inch vertical clearance with the ceiling.
   a. If the trucks are 8 feet wide and 12 feet tall, what is the smallest possible radius of the semicircle?
   b. What is the tallest 8-foot wide vehicle that can drive down the center of the tunnel?



not to scale

10. Graph the following system of inequalities and describe the graph.

$$\begin{cases} x^2 + y^2 \geq 4 \\ (x-5)^2 + (y-11)^2 \geq 16 \end{cases}$$

11. In sumo wrestling, participants wrestle in a *dohyo*, a circular arena with a 4.55-meter diameter centered inside a square with side length 6.7 meters. Pushing your opponent out of the circle is one way to win the match. Sometimes the area between the circle and square is layered with fine sand to assist in determining when a wrestler falls outside of the circle. Put the origin at the center of the arena and describe the sandy region with a system of 5 inequalities.

REVIEW

In 12 and 13, define the term. (Lessons 12-2, 12-1)

12. circle

13. parabola

14. Find an equation for the circle with center at (3, 6) and radius 2. (Lesson 12-2)

15. A circle with center at the origin passes through the point (5, 12). (Lesson 12-2)

a. Find the radius of the circle.

b. Find an equation for the circle.

16. Identify the focus and directrix of the parabola described by $y = \frac{1}{800}x^2$. (Lesson 12-1)

17. Expand and simplify. (Lessons 11-2, 6-1)

a. $(x + 8)^2 + y^2$ b. $(x + y)^2 + 8^2$ c. $(x + y + 8)^2$

18. A 3×5 drawing is enlarged to 6×10 by using a size change. (Lesson 4-4)

a. What is a matrix for the size change?

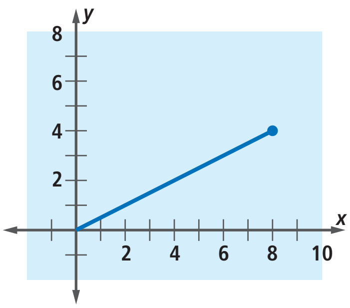
b. Suppose $A = (1, 2.3)$, $B = (0.3, 2)$, and $C = (2.7, 4.3)$ are three points on the smaller drawing. Write a matrix representing the locations of A', B', and C' on the image.

c. Find the distance between A and B.

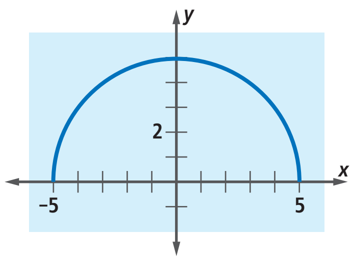
d. Find the distance between A' and B'.

In 19 and 20, the graph of a function is given. State a. the function's domain, and b. its range. (Lesson 1-4)

19.



20.

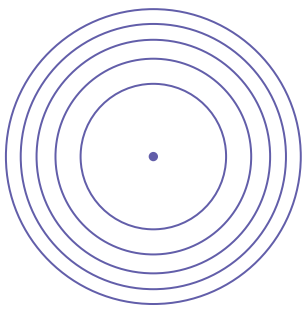


21. Caleb has three scraps of wood with lengths 1.5 feet, 2.3 feet, and 4 feet. Without cutting them, can he use these scraps as the three sides of a triangular flower planter? Why or why not? (Previous Course)

EXPLORATION

22. a. A target with five circles all with center (0, 0) is shown at right. If the largest circle has radius 1 and the areas of the five nonoverlapping regions are all equal, find the radii and equations for the five circles.

b. Generalize Part a.



QY ANSWER

$y = -\sqrt{9 - x^2} + 8$