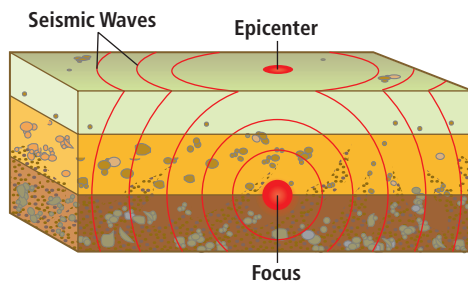## Lesson

12-2

Circles

▶ **BIG IDEA** From the geometric definition of a circle, an equation for any circle in the plane can be found.

As you know, a **circle** is the set of all points in a plane at a given distance (its **radius**) from a fixed point (its **center**). When a person throws a pebble into a calm body of water, *concentric circles* soon form around the point where the pebble hit the water. **Concentric circles** have the same center, but different radii. When an earthquake or mass movement of land (such as a volcanic eruption) occurs, various kinds of *seismic waves* radiate in roughly concentric circles from the *focus*, the point below Earth's surface where the earthquake began.

After reaching the surface, seismic waves travel along the ground in concentric circles around the *epicenter*, or the point on Earth's surface above the focus. Warning systems for tsunamis and other natural disasters are triggered by measuring seismic waves with an instrument called a *seismograph*.

Seismic Waves   **Epicenter**

Focus

Mental Math

Suppose Aaron throws a rock off a cliff and its height _h_ in meters _t_ seconds after he throws it is given by $h = -4.9t^2 + 4.5t + 25.$

a. Did Aaron throw the rock upward or downward?

b. What was the rock's initial velocity?

c. How high is the cliff?

d. Will the rock have hit the ground after 10 seconds?

Seismic waves travel at the same speed in all directions. For earthquakes on land, the fastest seismic waves, called *P-waves* or *compression waves*, can travel at speeds of up to $8 \frac{\text{km}}{\text{sec}}$. So the points on Earth's surface that a compression wave reaches at a given time make an approximate circle whose center is the epicenter of a quake.

Equations for Circles

You can find the equation for any circle using the definition of circle and the Pythagorean Distance Formula.

Example 1

Suppose the compression waves of an earthquake travel at a speed of $8 \frac{km}{sec}$. Find an equation for the set of points that are reached by the compression waves in 7 seconds.

Solution Let the unit of a graph equal one kilometer, and the origin $(0, 0)$ represent the epicenter of the quake. In 7 seconds, a compression wave travels about $8 \frac{km}{sec}$ ⋅ 7 sec = 56 km. So, the circle has radius 56.

Let (x, y) be any point on the circle. By the definition of circle, the distance between (x, y) and $(0, 0)$ is 56.

Use the Pythagorean Distance Formula.

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 56$$
$$(x - 0)^2 + (y - 0)^2 = 56^2 \quad \text{Square both sides.}$$
$$x^2 + y^2 = 3136 \quad \text{Simplify.}$$
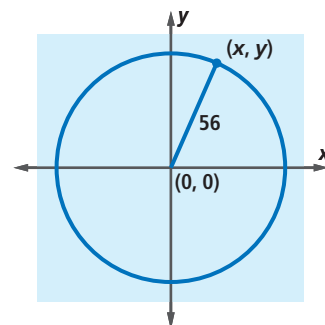
The equation you found in Example 1 is a quadratic equation in x and y. This is also true of equations for circles not centered at the origin.
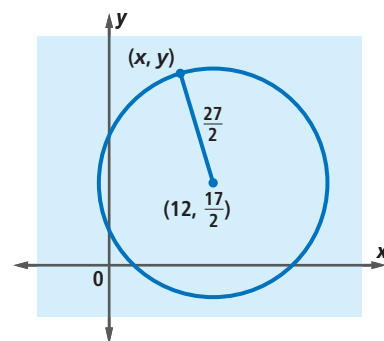
GUIDED

Example 2

Find an equation for the circle with radius $\frac{27}{2}$ and center $\left(12, \frac{17}{2}\right)$.
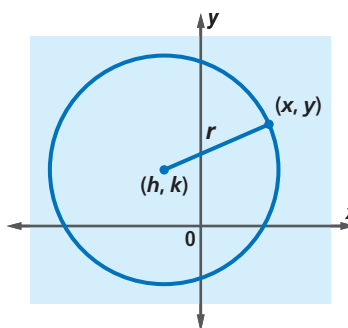
Solution Let (x, y) be any point on the circle.

Use the Distance Formula. $\qquad \sqrt{(x - \underline{})^2 + (y - \underline{})^2} = \underline{}$

Square both sides. $\qquad (x - \underline{})^2 + (y - \underline{})^2 = \underline{}$

STOP QY

Example 2 can be generalized to determine an equation for *any* circle. Let (h, k) be the center of a circle with radius r, and let (x, y) be any point on the circle. Then, by the definition of a circle, the distance between (x, y) and (h, k) equals r.

▶ **QY**

The point $(25.5, 8.5)$ satisfies the equation in Example 2. Check your answer to Example 2 by showing that this point is $\frac{27}{2}$ units from the center of the circle.

By the Distance Formula, $\sqrt{(x-h)^2 + (y-k)^2} = r$.

Squaring gives an equation without radicals: $(x-h)^2 + (y-k)^2 = r^2$.

This proves the following theorem.

> ### Circle Equation Theorem
>
> The circle with center (h, k) and radius r is the set of points (x, y) that satisfy $(x-h)^2 + (y-k)^2 = r^2$.

When the center of a circle is the origin, $(h, k) = (0, 0)$ and the equation becomes

$$x^2 + y^2 = r^2.$$

Graphing a Circle

By the Graph-Translation Theorem, the translation $T_{h,k}$ maps the circle with equation $x^2 + y^2 = r^2$ onto a circle with equation $(x-h)^2 + (y-k)^2 = r^2$.

Example 3

a. Find the center and radius of the circle with equation $(x+4)^2 + (y-5)^2 = 49$.

b. Graph this circle.

c. What translation maps the circle with equation $x^2 + y^2 = 49$ onto the circle you drew in Part b?

Solution

a. By the Circle Equation Theorem, the center $(h, k) = (-4, 5)$ and the radius $r = \sqrt{49} = 7$.

b. You can make a quick sketch of this circle by locating the center and then four points on the circle whose distance from the center is 7, as illustrated at the right.

c. The preimage circle is centered at the origin and the image circle is centered at $(-4, 5)$, so $T_{-4,5}$ translates $x^2 + y^2 = 49$ onto $(x+4)^2 + (y-5)^2 = 49$.
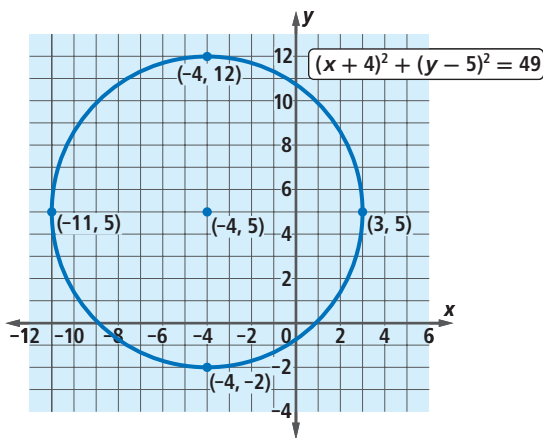
In Example 3, you graphed the circle by hand. How do you graph it on a graphing utility? A circle is not the graph of a function, and most graphing utilities will only graph functions. However, you can divide the circle in half with a horizontal line. Each half-circle, or **semicircle**, is the graph of a function, and both can be graphed in the same window to form a circle.
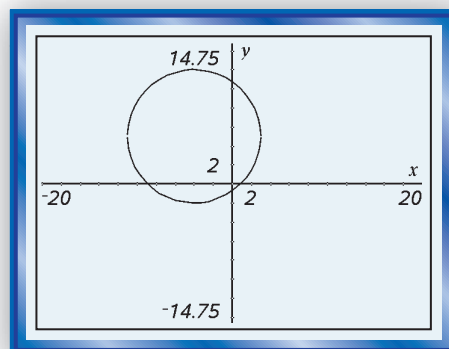
Example 4

a. Solve $(x + 4)^2 + (y - 5)^2 = 49$ for y.

b. Graph the circle with the equation in Part a on a graphing utility.

Solution

a. Solve for y on a CAS. One CAS solution is shown below.

$$\text{solve}\left((x+4)^2+(y-5)^2=49,y\right)$$
$$y=\left(\sqrt{-x^2-8\cdot x+33}-5\right) \text{ or } y=\sqrt{-x^2-8\cdot x+3?}$$

b. Copy the two equations into a graphing utility. Graph both on the same screen, using a square window. The graph of each equation is half of a circle, but due to the graphing utility's limitations, the circle's graph is not completely accurate.

If you know an equation for a circle and one coordinate of a point on the circle, you can determine the other coordinate of that point.

GUIDED

Example 5

Refer to the circle in Examples 3 and 4. Find the y-coordinate of each point with x-coordinate 1.

Solution Substitute __?__ for x in the equation for the circle $(x + 4)^2 + (y - 5)^2 = 49$ and solve for y.

$$(\underline{} + 4)^2 + (y - 5)^2 = 49 \qquad \text{Substitute.}$$

$$(y - 5)^2 = \underline{} \qquad \text{Simplify. Then subtract } \underline{} \text{ from both sides.}$$

$$|\underline{}| = \sqrt{\underline{}} \qquad \text{Take the square root of both sides and apply the Absolute Value-Square Root Theorem.}$$

$$y - \underline{} = \underline{} \text{ or } y - \underline{} = \underline{} \qquad \text{Definition of absolute value}$$

$$y = \underline{} \quad \text{or} \quad y = \underline{} \qquad \text{Add } \underline{} \text{ to both sides of each equation.}$$

Check Solve the original equation for the circle on a CAS for y such that x = 1. One solution is shown at the right. If yours looks different, check to see that it is equivalent to this one.

$$\text{solve}\left((x+4)^2+(y-5)^2=49,y\right)|x=1$$
$$y=\left(2\cdot\sqrt{6}-5\right) \text{ or } y=2\cdot\sqrt{6}+5$$
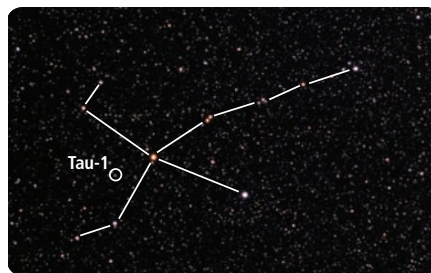
Questions

COVERING THE IDEAS

1. Suppose the epicenter for an earthquake is at the origin. If 1 unit on the graph represents 1 km, find an equation describing the set of points (x, y) reached by compression waves in 1 minute if the waves travel at $8\frac{km}{sec}$.

2. Repeat Question 1 if the epicenter is at $(40, 22)$.

In 3 and 4, consider the circle with equation $x^2 + y^2 = 34^2$.

3. What is the radius of this circle?

4. Tell whether the point is on the circle.
   - **a.** $(0, 0)$ **b.** $(-34, 0)$ **c.** $(0, 34)$ **d.** $(\sqrt{34}, 0)$

5. Write equations for two circles that are concentric and centered at the origin.

6. **Fill in the Blanks** The circle with equation $(x - h)^2 + (y - k)^2 = r^2$ has center __?__ and radius __?__.

In 7 and 8, an equation for a circle is given. State the center and radius of the circle, sketch the circle, and solve the equation for y.

7. $x^2 + y^2 = 121$

8. $(x + 7)^2 + (y + 3)^2 = 25$

9. Refer to the circle from Example 2. Find the y-coordinates of all points on the circle
   - **a.** where $x = -6$.   **b.** where $x = 800$.

10. Find an equation for the circle with center $\left(-\frac{1}{2}, \frac{5}{2}\right)$ and radius 8.

APPLYING THE MATHEMATICS

11. Planets in our solar system travel in elliptical orbits about the Sun, but these orbits are nearly circular. In the late 1990s, as astronomers searched for solar systems similar to ours, they discovered other planets with almost circular orbits. One example, discovered in 2002, is a planet orbiting the star Tau Gruis. This planet is 2.5 astronomical units (AU) away from Tau Gruis. (1 AU is the average distance between Earth and the Sun.)

    Tau Gruis, visible from the Southern Hemisphere, is located about 100 light-years from Earth.

    - **a.** Write an equation for the orbit of this planet about Tau Gruis.
    - **b.** If Earth's orbit is assumed to be circular, it revolves about the Sun at a speed of approximately 67,062 miles per hour. If it takes 3.5 years for the planet orbiting Tau Gruis to make one revolution, about how fast is the planet moving?

12. The equation of the circle in Example 4 is equivalent to an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. What are the values of $A, B, C, D, E,$ and F?

13. A circle centered at the origin has radius 5.
    a. Find the equation of its image under the translation $T_{4, -1}$.
    b. Identify one point on the circle. Verify that its image satisfies the image circle of Part a.

14. To locate the epicenter of an earthquake, data about waves originating from the epicenter are collected by earthquake research stations. Three circles, each centered at a research station with a radius equal to the distance from the epicenter to the particular station, allow one to determine the epicenter location. The location is given by a unique intersection shared by all three circles. Suppose that:

    Station 1 is 7 miles from the epicenter and located at (6, 1).
    Station 2 is 5 miles from the epicenter and located at (11, 8).
    Station 3 is 4 miles from the epicenter and located at (2, 8).

    a. Graph the three circles defined above.
    b. Find the epicenter of the earthquake.
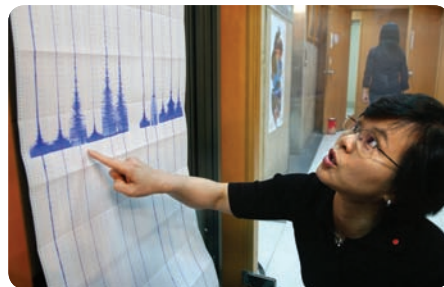
Earthquake data allow a research station to calculate its distance from an epicenter, but not its direction.

REVIEW

15. A parabola has focus (0, –1) and directrix $y = 1$. (Lesson 12-1)
    a. What is its vertex? Is it a minimum point or a maximum point?
    b. Give an equation for the parabola.
    c. Give an equation for its axis of symmetry.

16. Give the focus and directrix of the parabola $y = \frac{2}{3}x^2$. (Lesson 12-1)

17. What are the zeros of the 4th-degree polynomial function graphed at the right? (Lesson 11-4)

18. In 2007, Karla deposited $5000 in a retirement account paying interest compounded continuously. If no additional deposits or withdrawals are made, when Karla retires in 2040 the account will be worth $8172.40. What is the annual percentage yield of this account? (Lessons 9-10, 9-1, 7-4)

19. Find an equation for the line containing the origin and (–3, 7). (Lesson 3-4)
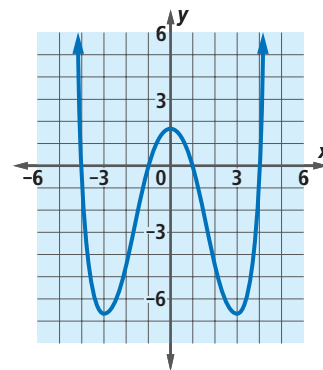
EXPLORATION

20. A *lattice point* is a point with integer coordinates. If possible, find an equation for a circle that passes through
    a. no lattice points.
    b. exactly one lattice point.
    c. exactly two lattice points.
    d. exactly three lattice points.
    e. more than ten lattice points.

QY ANSWER

$\left(12, \frac{17}{2}\right)$ and $\left(\frac{51}{2}, \frac{17}{2}\right)$ are on a horizontal line
$\frac{51}{2} - 12 = \frac{27}{2}$ units apart.