Lesson
12-1 Parabolas

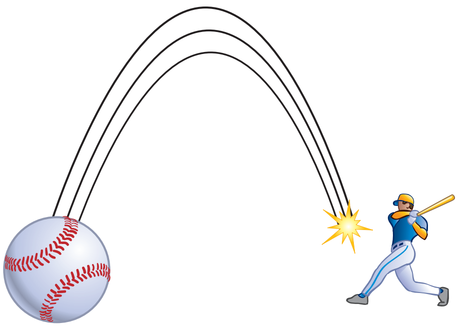
Vocabulary

parabola
focus, directrix
axis of symmetry
vertex of a parabola
paraboloid

BIG IDEA From the geometric definition of a parabola, it can be proved that the graph of the equation $y = ax^2$ is a *parabola*.

What Is a Parabola?

In Chapter 6 you were told that the path of a shot or tossed object, such as a fly ball in baseball, is part of a *parabola*.



But how do we know this? In order to determine whether a curve is a parabola, a definition of *parabola* is necessary. Here is a geometric definition.

Mental Math

Which expression or equation is not equivalent to the others?

a. $\log_{10} 1742, \log_{16} 1742, \log 1742$

b. $\log_3 100, 2 \log_3 10, \log_3 20$

c. $r = e^{3t}, r = 3e^t, \ln r = 3t$

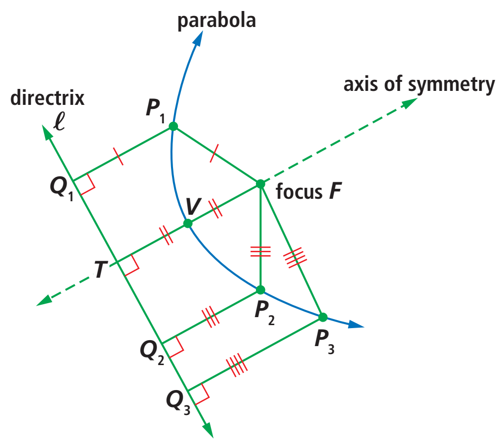
d. $\frac{\ln 2}{\ln p}, \frac{\log 2}{\log p}, \frac{\log_6 2}{\log_6 p}$

Definition of Parabola

Let ℓ be a line and F be a point not on ℓ. A **parabola** is the set of all points in the plane of ℓ and F equidistant from F and ℓ.

To understand the definition of parabola, recall that the distance from a point P to a line ℓ is the length of the perpendicular from P to ℓ. In the diagram at the right below, four points on a parabola, V, P_1, P_2, and P_3, are identified. Note that each is equidistant from F and the line ℓ. For example, $\overline{P_1Q_1} \perp \ell$ and $P_1Q_1 = P_1F$. Also, $\overline{P_2Q_2} \perp \ell$ and $P_2Q_2 = P_2F$, and so on.

F is the **focus** and ℓ is the **directrix** of the parabola. Thus, a parabola is the set of points in a plane equidistant from its focus and its directrix. Neither the focus nor directrix is on the parabola. The line through the focus perpendicular to the directrix is the **axis of symmetry** of the parabola. The point V on the parabola and on the axis of symmetry is the **vertex of the parabola**.



STOP QY1

Drawing a Parabola

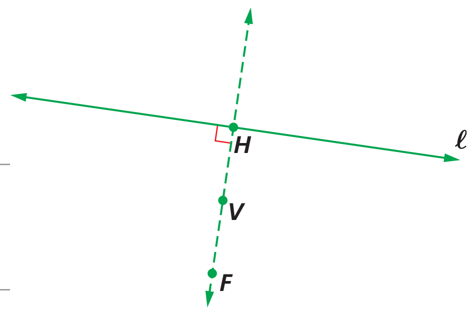
You can draw as many points on a parabola as you wish using only a compass and a straightedge.

> **QY1**
>
> FV is equal to what other distance shown in the diagram on the previous page?

Activity

MATERIALS compass, straightedge

Step 1 Begin with a blank sheet of paper. Draw a directrix ℓ and a focus F not on ℓ. With H on ℓ, construct $\overleftrightarrow{FH}$ perpendicular to ℓ. Then construct the midpoint of $\overline{FH}$ and label it V for vertex. Your drawing should resemble the one at the right.



Step 2 Let $FH = d$. Construct a line segment parallel to ℓ of length $2d$ through F, where F is the midpoint. Label the endpoints of this segment P_1 and P_2.

Step 3 **a.** Construct perpendicular line segments from P_1 to ℓ and P_2 to ℓ. Note that P_1 and P_2 are vertices of squares with $\overline{FH}$ as the common side.

b. How does FP_1 compare to the perpendicular distance from P_1 to ℓ? What does this tell you about P_1?

c. How does FP_2 compare to the perpendicular distance from P_2 to ℓ? What does this tell you about P_2?

Step 4 Construct a circle with center F and any radius $r > \frac{d}{2}$.

Step 5 **a.** Construct a line n parallel to ℓ that is distance r from ℓ. (*Hint:* Use your compass to measure r.) Label the two intersections of this line and the circle P_3 and P_4.

b. Are P_3 and P_4 on the parabola? Justify your answer.

Step 6 Find two more points by repeating Steps 4 and 5 for a different $r > \frac{d}{2}$.

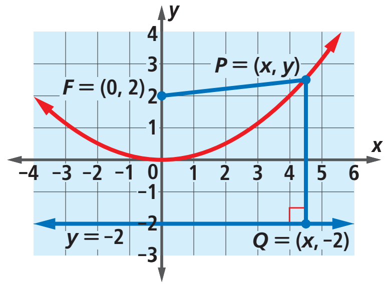
Step 7 You have found seven points on the parabola. Connect them with a smooth curve to sketch part of a parabola.

Equations for Parabolas

Suppose that you know the coordinates of the focus and an equation for the directrix of a parabola. You can find an equation for the parabola by using the definition of parabola and the Pythagorean Distance Formula.

Example

Find an equation for the parabola with focus $F = (0, 2)$ and directrix $y = -2$.



Solution Sketch the given information. Let $P = (x, y)$ be any point on the parabola. Because the directrix is a horizontal line, the distance from a point on the parabola to the directrix is measured along a vertical line. Let Q be the point on the directrix and on the vertical line through P. If $P = (x, y)$, then $Q = (x, -2)$.

$PF = PQ$	definition of parabola
$\sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(x-x)^2 + (y-(-2))^2}$	Pythagorean Distance Formula
$x^2 + (y-2)^2 = (y+2)^2$	Square both sides.
$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$	Expand.
$x^2 - 4y = 4y$	Add $-y^2 - 4$ to both sides.
$x^2 = 8y$	Add $4y$ to both sides.
$y = \frac{1}{8}x^2$	Solve for y.

An equation for the parabola is $y = \frac{1}{8}x^2$.

Check Pick any point on $y = \frac{1}{8}x^2$. We use $A = (12, 18)$. Now show that A is equidistant from $F = (0, 2)$ and $y = -2$.

$$AF = \sqrt{(12-0)^2 + (18-2)^2} = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$$

The distance from A to $y = -2$ is the distance from $(12, 18)$ to $(12, -2)$, which is 20, also. So, A is on the parabola with focus $(0, 2)$ and directrix $y = -2$.

In the Example, if you replace the focus by $\left(0, \frac{1}{4}\right)$ and replace the directrix by $y = -\frac{1}{4}$, the equation for the parabola is $y = x^2$. If a is nonzero, $(0, 2)$ is replaced by $\left(0, \frac{1}{4a}\right)$, and the directrix is replaced by $y = -\frac{1}{4a}$, then the parabola has equation $y = ax^2$. The derivation of both of these equations uses the same steps as the Example, and demonstrates the following theorem.

Focus and Directrix of a Parabola Theorem

For any nonzero real number a, the graph of $y = ax^2$ is the parabola with focus at $\left(0, \frac{1}{4a}\right)$ and directrix at $y = -\frac{1}{4a}$.

STOP **QY2**

Recall that because the image of the graph of $y = ax^2$ under the translation $T_{h,k}: (x, y) \rightarrow (x + h, y + k)$ is the graph with equation $y - k = a(x - h)^2$, the graph of any quadratic equation of the form $y = a(x - h)^2 + k$ or $y = ax^2 + bx + c$ is also a parabola. You can find the focus and directrix of a parabola with equation $y = a(x - h)^2 + k$ by applying the appropriate translation to the focus and directrix of $y = ax^2$.

When $a < 0$, you have learned that the parabola opens down. In this case, when the vertex is (0, 0), the directrix is above the x-axis, and the focus is below.

If a parabola is rotated in space around its axis of symmetry it creates a 3-dimensional **paraboloid**. The focus of a paraboloid is the focus of the rotated parabola. Paraboloids are common in modern technology. The shape of a satellite receiving dish is based on a paraboloid. Residents of a wheat-growing commune in southern China use a tiled solar reflector in the shape of a paraboloid. A teapot is placed at the focus of the paraboloid. Sunlight is reflected toward the teapot, boiling the water in 20 minutes without burning any wood, which is a precious resource. Cooking with a Dutch oven, as shown at the right, follows the same principle.

▸ QY2

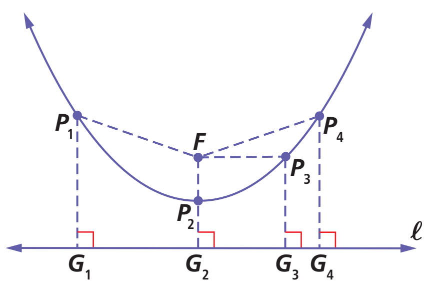
Find the focus and directrix of the parabola with equation $y = -\frac{1}{6}x^2$.

Questions

COVERING THE IDEAS

1. a. Can the focus of a parabola be a point on the directrix? Why or why not?
   b. Can the vertex be on the directrix? Why or why not?

True or False In 2–4, refer to the parabola at the right with focus F and directrix ℓ. P_1, P_2, P_3, and P_4 are points on the parabola.



2. $P_3F = FG_3$
3. $FG_1 = FG_2$
4. The focus of this parabola is its vertex.

5. Refer to the Example.
   a. Graph the parabola with equation $y = \frac{1}{8}x^2$.
   b. Name its focus, vertex, and directrix.
   c. Verify that the point (5, 3.125) is equidistant from the focus and directrix, and therefore is a point on the parabola $y = \frac{1}{8}x^2$.
   d. Find another point on the graph of $y = \frac{1}{8}x^2$. Show that it is equidistant from the focus and directrix.

6. Verify that the graph of $y = -x^2$ is a parabola with focus $\left(0, -\frac{1}{4}\right)$ and directrix $y = \frac{1}{4}$ by choosing a point on the graph and showing that two appropriate distances are equal.

7. Let $F = (0, -3)$ and ℓ be the line with equation $y = 3$. Write an equation for the set of points equidistant from F and ℓ.

8. a. Using graph paper, follow the steps of the Activity to draw five points that are on the parabola with focus $F = (0, 1)$ and directrix defined by $y = -1$. You do not need to construct with a ruler and compass.
   b. Refer to the Example. Find an equation for the parabola you drew in Part a.
   c. Verify that the points you drew in Part a are on the graph of the function defined in Part b.

9. Give the focus and directrix of the parabola with equation $y = -2xz^2$.

10. What is a paraboloid?

APPLYING THE MATHEMATICS

11. What are the focus and directrix of $y - 6 = (x + 5)^2$?

12. Prove the Focus and Directrix of a Parabola Theorem.

In 13 and 14, an equation for a parabola is given.
   a. Tell whether the parabola opens up or down.
   b. Give the focus of the parabola.

13. $y = -5x^2$

14. $y = \frac{1}{4}x^2$

15. a. Find an equation for the parabola with focus (5, 0) and directrix $x = -5$.
   b. Give the coordinates of three points on this parabola, including the vertex.

The Municipal Asphalt Plant in New York City has a parabolic shape. This landmark facility is used for community sports, fitness, and recreation.

REVIEW

16. An isotope has a half-life of 135 seconds. How long will it take 75 mg of this isotope to decay to 20 mg? **(Lesson 9-2)**

17. Solve for y. **(Lessons 8-8, 6-2)**

a. $y^2 = 13$ **b.** $(y + 3)^2 = 13$

c. $\sqrt{y} = 13$ **d.** $\sqrt{y + 3} = 13$

18. Determine whether the set of points (x, y) satisfying the given equation describes y as a function of x. Explain your answer. **(Lessons 8-2, 2-5)**

a. $y = (x + 1)^2$ **b.** $x = (y + 1)^2$

19. Simplify. **(Lessons 7-8, 7-7)**

a. $\left(\frac{16}{49}\right)^{\frac{1}{2}}$ **b.** $(0.0001)^{-\frac{3}{4}}$

20. Suppose the transformation $T_{-3,2}$ is applied to the parabola with equation $y = \frac{5}{9}x^2$. Find an equation for its image. **(Lesson 6-3)**

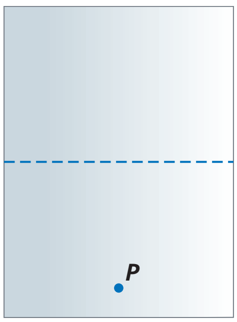
21. Fill in the Blank If x is a real number, then $\sqrt{x^2} =$ __?__. **(Lesson 6-2)**

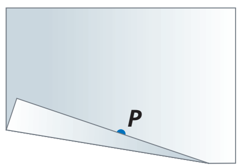
A x **B** $-x$ **C** $|x|$ **D** none of these

EXPLORATION

22. Parabolas can be formed without equations or graphs. Follow these steps to see how to make a parabola by folding paper.

a. Start with a sheet of unlined paper. Fold it in half as shown at the right. Cut or tear along the fold to make two congruent pieces. On one piece mark a point P about one inch above the center of the lower edge. Fold the paper so that the lower edge touches P, and crease well as shown at the right. Then unfold the paper. Repeat 10 to 15 times, each time folding so that a different point on the bottom edge of the paper aligns with P. The creases represent the *tangents* to a parabola. (A *tangent* to a parabola is a line not parallel to its line of symmetry that intersects the parabola in exactly one point.) Where are the focus and directrix of this parabola?





b. On the other piece of paper mark a point Q approximately in the center. Repeat the procedure used in Part a. Where are the focus and directrix for this parabola?

c. Fill in the Blank The two parabolas formed in Parts a and b illustrate the property that as the distance between the focus and directrix increases, the parabola __?__.

QY ANSWERS

1. VT

2. focus: $(0, -24)$; directrix: $y = 24$