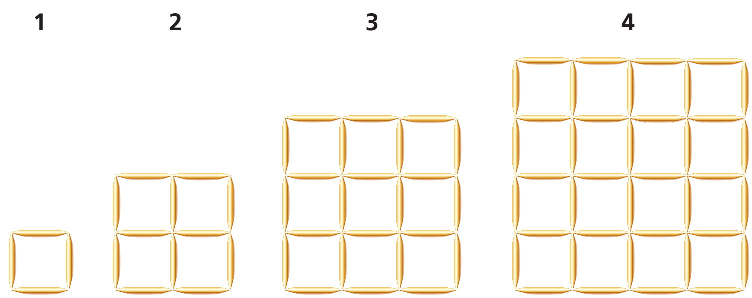
Lesson 2-5 Testing Equivalence

BIG IDEA You can test whether algebraic expressions are equivalent using substitution, tables, graphs, or properties.

One of the most powerful aspects of algebra is that it offers many different approaches to solving problems. It may seem to you that learning different ways to do the same thing is a waste of time, but it is not. The knowledge of different approaches increases your ability to undertake new and different situations or problems on your own.

In Chapter 1, you began working with one of the most important ideas in algebra: *equivalence.* This idea will be examined and reexamined throughout your study of mathematics. Numbers, graphs, and algebraic properties give us three different approaches for testing equivalence.

In Lesson 1-3, you were asked to write expressions to describe the number of toothpicks used to form patterns, as shown below.



Suppose Yolanda and Suki created the following expressions to represent the number of toothpicks in the nth term.

Yolanda: $4n + 2n(n - 1)$

Suki: $n(n + 1 + n + 1)$

How can you tell if these expressions produce the same result for each value of n? You have seen a variety of methods used in this chapter. Some methods are more powerful than others.

Mental Math

Estimate to the nearest integer.

a. $\frac{4}{5} + 2\frac{1}{10} + 7\frac{7}{8}$

b. $\frac{4}{5} - 2\frac{1}{10} + 7\frac{7}{8}$

c. $\frac{4}{5} - 2\frac{1}{10} - 7\frac{7}{8}$

Example 1

Test whether $4n + 2n(n - 1)$ and $n(n + 1 + n + 1)$ are equivalent.

Method 1: Use numbers to test for equivalence. Choose $n = 1$. Evaluate each expression.

If $n = 1$, then $4n + 2n(n - 1) = 4 \cdot 1 + 2 \cdot 1(1 - 1) = 4$.

If $n = 1$, then $n(n + 1 + n + 1) = 1(1 + 1 + 1 + 1) = 4$, the number of toothpicks in the 1st term.

There are 12 toothpicks in the second term. Do both expressions have the value 12 when $n = 2$?

If $n = 2$, then $4n + 2n(n - 1) = 4 \cdot 2 + 2 \cdot 2(2 - 1) = 12$.

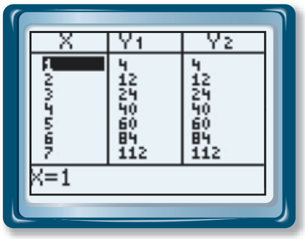
If $n = 2$, then $n(n + 1 + n + 1) = 2(2 + 1 + 2 + 1) = 12$, which checks.

You can repeat this process to test many numbers.

Technology can allow you to test many numbers at once, as seen in Method 2.

Method 2: Use tables to test equivalence. A graphing calculator can quickly make a table. Substitute x for n when entering the expressions into the calculator.

For Yolanda's expression enter into Y1:
$4x + 2x(x - 1)$.
For Suki's expression enter into Y2:
$x(x + 1 + x + 1)$.
We set a table starting at 1 with increments of 1.

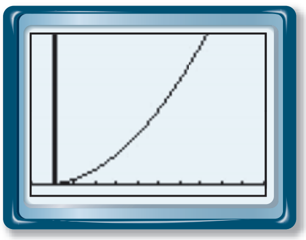


The seven numbers located in the left column are each substituted into the two expressions. The results are automatically calculated and displayed in the second and third columns. In each case, the values of the expressions are equal, and they equal the numbers of toothpicks.

You can scroll up or down the left column of the table to see more values. So for these values of n, the expressions $4n + 2n(n - 1)$ and $n(n + 1 + n + 1)$ have the same value.

STOP **QY**

Method 3: Use graphs to test for equivalence. Use the formulas $Y1 = 4x + 2x(x - 1)$ and $Y2 = x(x + 1 + x + 1)$. The table helps you decide on a viewing window. One window that fits the ordered pairs listed is $-1 \le x \le 10$, $0 \le y \le 120$.



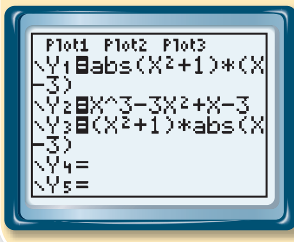
QY

Two of the three expressions below are equivalent. Use the table feature on the graphing calculator to determine which are equivalent. Use at least three rows of your table to explain your answer. The screen below shows how the equations should be entered into the calculator.

a. $|x^2 + 1| \cdot (x - 3)$

b. $x^3 - 3x^2 + x - 3$

c. $(x^2 + 1) \cdot |x - 3|$



The graphs of the two expressions seem to be identical. But, even on a calculator, only a limited number of points are actually being graphed. How can you be sure the expressions are always equal? The answer is to use the properties of operations that are true for all real numbers.

Method 4: Use properties to test equivalence.

$4n + 2n(n - 1)$	$= 4n + 2n^2 - 2n$	Expand the expression.
	$= 2n^2 + 2n$	Combine like terms.
$n(n + 1 + n + 1)$	$= n(2n + 2)$	Combine like terms.
	$= 2n^2 + 2n$	Expand the expression.

In each case, the result is $2n^2 + 2n$. Since both expressions are equal to the same third expression, they are equal to each other (by the Transitive Property of Equality). Therefore, for any value of n, $4n + 2n(n - 1) = n(n + 1 + n + 1)$.

Properties of operations are powerful because they can show that a pattern is true for all real numbers. But the other methods are useful too. Testing specific numbers, either by hand or in a table, can help you decide if two expressions *seem* equivalent. These methods can often help you detect a counterexample. Testing numbers is also a good way to catch your own mistakes.

GUIDED

Example 2

A common error that some students make is to think that $4x - x$ is equivalent to 4 for all values of x. Here are three ways to show that these expressions are not equivalent.

Solution

Method 1: Substitute a value for x to show that $4x - x$ is not equal to 4.

Method 2: Graph Y1 $= 4x - x$ and Y2 $= 4$. Are the graphs identical?

Method 3: Create a table of values for Y1 $= 4x - x$ and Y2 $= 4$.

You should find that for almost all values of x, the expressions do not have the same values. Therefore, $4x - x$ is not equivalent to 4. You could also simplify $4x - x$ to $3x$, which clearly is not 4 for every value of x.

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Example 3

Are $-x^2$ and $(-x)^2$ equivalent expressions? If so, explain. If not, provide a counterexample.

Solution Pick a value for x. Suppose you pick 6. Then $-x^2 = -6^2$ and $(-x)^2 = (-6)^2$.

For -6^2, follow the order of operations and square 6 *before* taking the opposite.

If $x = 6$, then $-x^2 = \underline{\ ?\ }$	Substitute.
$= -(\underline{\ ?\ })(\underline{\ ?\ })$	Evaluate powers first.
$= \underline{\ ?\ }$	Simplify.

For $(-6)^2$, square -6.

If $x = 6$, then $(-x)^2 = \underline{\ ?\ }$	Substitute.
$= (\underline{\ ?\ })(\underline{\ ?\ })$	Evaluate powers first.
$= \underline{\ ?\ }$	Simplify.

Because $\underline{\ ?\ }$ and $\underline{\ ?\ }$ are not equal, the expressions are not equivalent. Therefore, 6 is a counterexample.

Guided Example 3 used just one counterexample to show that $-x^2$ and $(-x)^2$ are not equivalent. Only one is necessary. However, to show that expressions *are* equivalent requires much more. You must show equality for *every* possible value of the variable. This is why using properties that apply to all numbers is so important.

Example 4

Use properties to show that $(2x^2 + 3) - 8(3x^2 + 4)$ is equivalent to $1 - 27x^2 - 30 + 5x^2$.

Solution Simplify each expression and check that the results are the same. Expand and combine like terms.

$(2x^2 + 3) - 8(3x^2 + 4) = 2x^2 + 3 - 24x^2 - 32$	Distributive Property
$= -22x^2 - 29$	Combine like terms.

Now simplify $1 - 27x^2 - 30 + 5x^2$.

$1 - 27x^2 - 30 + 5x^2 = -22x^2 - 29$	Combine like terms.

Because both $(2x^2 + 3) - 8(3x^2 + 4)$ and $1 - 27x^2 - 30 + 5x^2$ equal $-22x^2 - 29$, we can say that

$(2x^2 + 3) - 8(3x^2 + 4) = 1 - 27x^2 - 30 + 5x^2$ by the Transitive Property of Equality.

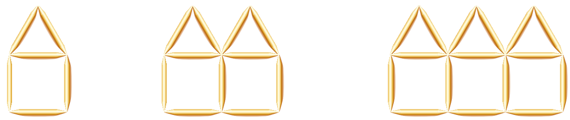
Questions

COVERING THE IDEAS

1. Refer to the sequence of toothpicks shown below and the table at the right.

Term Number	Number of Toothpicks
1	6
2	11
3	16

1 2 3



   a. How many toothpicks would be used to make the 4th term?

   b. Dion and Ellis wrote the expressions below to give the number of toothpicks used to make the nth term. Test to see if these two expressions are equivalent for the values $n = 4$, 5, and 6.

   Dion: $1 + 3n + 2n$ Ellis: $6n - (n - 1)$

   c. Use graphs with $0 \leq n \leq 10$ to test whether the two expressions in Part b are equivalent.

   d. Simplify each expression in Part b to test whether they are equivalent.

In 2–4, test the two given expressions for equivalence by using a table or graph, or by simplifying the expressions.

2. $4n - 15$ and $4(n - 4) - 1$

3. $3x^2 + 6x(x + 2)$ and $3x^2 + 6x^2 + 2$

4. $(5 + x)^2$ and $25 + 10x + x^2$

5. Is $3x^2$ equivalent to $(3x)^2$? If so, explain. If not, provide a counterexample.

APPLYING THE MATHEMATICS

6. Consider the expressions $x \cdot x$ and $2x$.

   a. Copy and complete the table of values at the right.

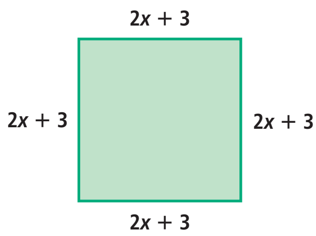
   b. Give two values of x for which $x \cdot x = 2x$.

   c. Give two values of x for which $x \cdot x$ does *not* equal $2x$.

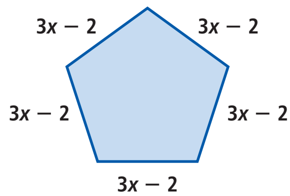
   d. Graph $y = x \cdot x$ and $y = 2x$. Circle the points that correspond to your answer from Part b.

x	$x \cdot x$	$2x$
–3	?	?
–2	?	?
–1	?	?
0	?	?
1	?	?
2	?	?
3	?	?

7. The perimeter of the square below can be written as $2x + 3 + 2x + 3 + 2x + 3 + 2x + 3$, or $4(2x + 3)$. Verify the two expressions are equivalent by using a table or graph, or simplifying each expression.



8. Write two equivalent expressions for the perimeter of the regular pentagon at the right.

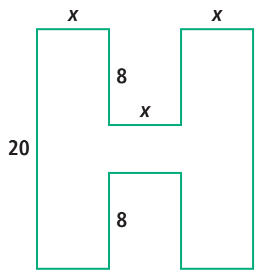


9. Manuel found the area of the "H" shape below to be $20(3x) - 2(8x)$, while Lina got $20x + 4x + 20x$. Are they equivalent?

   a. Use a table to find your answer.

   b. Use algebraic expressions to answer the question.

   c. Let $x = 5$. Find the area of the shape.



REVIEW

In 10–12, simplify the expression. (Lessons 2-4, 2-1)

10. $8b - 5(7b + 3)$

11. $-1.4(2v - 4) + 3.6v$

12. $\frac{12m + 2}{3} - \frac{6m - 1}{3}$

13. Determine whether each expression is equivalent to $-(18 - 5x)$. (Lesson 2-4)

   a. $-18 - (-5x)$ **b.** $-9(2 - 5x)$

   c. $-18 - 5x$ **d.** $-18 + 5x$

14. a. Pick any number and do the following number puzzle.

Step 1 Add 23 to your number.

Step 2 Double the sum.

Step 3 Subtract twice your original number.

Step 4 Add 3.

Step 5 Subtract 42.

You will be left with 7.

b. Let n be your original number. Develop an algebraic expression for each step of the puzzle. (**Lesson 2-3**)

15. ***Skill Sequence*** Simplify each expression. (**Lesson 2-2**)

a. $p \cdot p \cdot p$ b. $p + p + p$

c. $2p \cdot 2p \cdot 2p$ d. $2p + 2p + 2p$

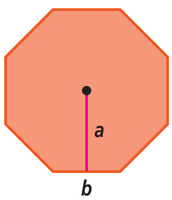
16. Use the Distributive Property to find $5 \cdot 999{,}999$ in your head. (**Lesson 2-1**)

In 17 and 18, give a counterexample to show that the two expressions are *not* equivalent. (Lesson 1-3)

17. $6 + m$ and $2m - 3(m - 2)$ 18. $\frac{y}{2} + \frac{3}{2}$ and $\frac{3y}{4}$

EXPLORATION

19. Two expressions used to calculate the area of the regular octagon below are $4ba$ and $\frac{1}{2}pa$, where p is the perimeter of the octagon. Explain how b and p are related and use this to demonstrate algebraically that the two expressions are equivalent.



QY ANSWER

Assume $Y1 = |x^2 + 1| \cdot (x - 3)$, $Y2 = x^3 - 3x^2 + x - 3$, and $Y3 = (x^2 + 1) \cdot |x - 3|$. From the table it appears that Y1 and Y2 are equivalent. We also see that neither Y1 nor Y2 is equivalent to Y3.

X	Y1	Y2	Y3
–1	–8	–8	8
0	–3	–3	3
1	–4	–4	4