

Lesson

1-8

Explicit Formulas
for Sequences

► **BIG IDEA** Sequences can be thought of as functions, but they have their own notation different from other functions. If their terms are real numbers, they are real functions and can be graphed.

Recognizing and Representing Sequences

In previous mathematics courses you have seen many sequences. A *sequence* is an ordered list of numbers or objects. Specifically, a **sequence** is defined as a function whose domain is the set of all positive integers, or the set of positive integers from a to b . Each item in a sequence is called a **term of the sequence**. In the following Activity, you will explore a sequence.

Activity

The collections of dots below form the first five terms of a sequence of triangular arrays. The numbers of dots in each collection form a sequence of numbers.



Step 1 Complete the table to show the number of dots in each of the terms pictured.

Step 2 Notice that after the first term of the sequence, each subsequent term adds a predictable and increasing number of dots to the previous term. Use this fact to complete the table for the next four terms.

Step 3 This process can be continued for as long as you want. You can even think of it as going on forever. Explain why the set of ordered pairs (term number, number of dots) describes a function.

Vocabulary

sequence
term of a sequence
subscript
index
explicit formula
discrete function

Mental Math

Let $a = -3$ and $b = 3$.
Evaluate.

- $a^2 - b^2$
- $b^2 - a^2$
- $(a - b)^2$
- $(b - a)^2$
- $(ab)^2$

Term Number	Number of Dots
1	
2	
3	
4	
5	

Term Number	Number of Dots
6	
7	
8	
9	

The sequence you explored in the Activity is the sequence of *triangular numbers*. This sequence defines a function whose domain is the set of all positive integers. If you call this function T , then $T(1) = 1$, $T(2) = 3$, $T(3) = 6$, A notation for sequences more common than $f(x)$ notation is to put the argument in a *subscript*. A **subscript** is a label that is set lower and smaller than regular text. Using subscripts, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, The notation $T_3 = 6$ is read “T sub three equals six.” The subscript is often called an **index** because it *indicates* the position of the term in the sequence.

STOP See Quiz Yourself 1 at the right.

▶ QUIZ YOURSELF 1

What are T_4 and T_5 ?

Writing Explicit Formulas for Sequences

Many sequences can be described by a rule called an **explicit formula** for the n th term of the sequence. Explicit formulas are important because they can be used to calculate any term in the sequence by substituting a particular value for n .

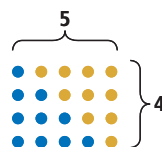
To find an explicit formula for the n th triangular number T_n , you can use the fact that the area of a triangle is half the area of a rectangle.

Notice that each triangular array of dots can be arranged to be half of a rectangular array.



For instance, the number of dots representing the 4th triangular number is half the number of dots in a 4 by 5 rectangular array.

$$T_4 = \frac{1}{2} \cdot 4 \cdot 5 = 10$$



You can generalize this idea to develop a formula for T_n .

Term Number	Value of Term (number of dots)
1	$T_1 = \frac{1}{2} \cdot 1 \cdot 2 = 1$
2	$T_2 = \frac{1}{2} \cdot 2 \cdot 3 = 3$
3	$T_3 = \frac{1}{2} \cdot 3 \cdot 4 = 6$
4	$T_4 = \frac{1}{2} \cdot 4 \cdot 5 = 10$
⋮	⋮
n	$T_n = \frac{1}{2} \cdot n \cdot (n + 1)$

The number of dots in the n th rectangle is $n(n + 1)$. T_n is half that.

$$T_n = \frac{1}{2} \cdot n \cdot (n + 1) = \frac{n(n + 1)}{2}$$

Thus an explicit formula for the number of dots in the n th term is

$$T_n = \frac{n(n + 1)}{2}.$$

STOP See Quiz Yourself 2 at the right.

▶ QUIZ YOURSELF 2

What is the 15th triangular number? The 100th triangular number?

Example 1

Suppose you flip a fair coin until it comes up tails. The probability that you will not have had an outcome of tails after n flips is given by the sequence

$$p_n = \left(\frac{1}{2}\right)^n.$$

- Compute and graph the first four terms of this sequence.
- Evaluate p_{20} , and explain what it represents.

Solution

- Substitute 1, 2, 3, and 4 for n in the formula and graph the ordered pairs (n, p_n) .

$$p_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$p_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$p_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

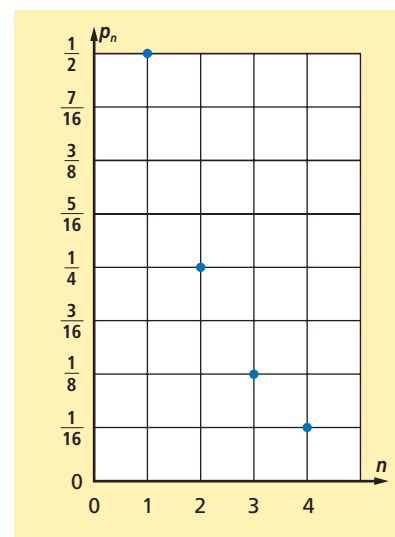
$$p_4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

- Substitute $n = 20$ into the formula.

$$p_{20} = \left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576}$$

p_{20} is the probability that you will not have had an outcome of tails after 20 flips.

STOP See Quiz Yourself 3 at the right.



▶ QUIZ YOURSELF 3

Write $p_n = \left(\frac{1}{2}\right)^n$ using function notation.

Using Explicit Formulas for Sequences

Sequences arise naturally in many situations in science, business, finance, and other areas. Example 2 looks at a sequence in finance.

GUIDED

Example 2

It is common for people to save money in savings accounts such as Certificates of Deposits (CDs) that yield a high interest rate paid once a year. Suppose you deposited \$28,700 and expected a 4.1% interest rate to be *compounded* annually. Then the formula $S_n = 28,700(1.041)^{n-1}$ gives your total savings at any time during the year leading up to the n th anniversary.

- Compute the first five terms of the sequence.
- Compute the hundredth term of the sequence.
- What does your answer to Part b mean in the context of this problem?

Solution 1

- Define the sequence using function notation on a CAS and compute the first five values.

$$S(1) = \underline{\quad ? \quad}$$

$$S(2) = \underline{\quad ? \quad}$$

$$S(3) = \underline{\quad ? \quad}$$

$$S(4) = \underline{\quad ? \quad}$$

$$S(5) = \underline{\quad ? \quad}$$

- Compute $S(100)$ in the same way.

$$S(100) = \underline{\quad ? \quad}$$

- This sequence gives the total savings at the end of the n th year. So, $S(100) = \underline{\quad ? \quad}$ means that on the 100th anniversary of the account opening, there will be $\underline{\quad ? \quad}$ in the account.

Solution 2

- Enter the formula into a grapher and generate a table to view the first five values.

The table start value is $n = \underline{\quad ? \quad}$.

The increment is $\underline{\quad ? \quad}$.

The table end value is $n = \underline{\quad ? \quad}$.

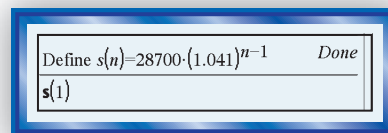
- Scroll down to see the value of $S(n)$ when $n = 100$.

$$S(100) = \underline{\quad ? \quad}$$

- After 100 years, there will be $\underline{\quad ? \quad}$ in the account.

CERTIFICATES OF DEPOSIT		
WEEK OF 10 24		
\$ 1000 MIN. DEPOSIT		
	RATE	ANNUAL YIELD
3 MONTHS	2.60%	2.63%
6 MONTHS	3.20%	3.25%
1 YEAR	3.94%	4.00%
18 MONTHS	4.67%	4.75%
2 YEARS	4.67%	4.75%
30 MONTH	4.67%	4.75%
3 YEARS	4.67%	4.75%
MONEY MARKET		2.50%
I R A		
18 MOS	3.85	4.31
24 & 30	4.45	4.52
36 MOS	4.45	4.52

Rates for Certificates of Deposit



x,n	
1.	
2.	
3.	
4.	
5.	

A sequence is an example of a *discrete function*. A **discrete function** is a function whose domain can be put into one-to-one correspondence with a finite or infinite set of integers, with gaps, or intervals, between successive values in the domain. The graphs of discrete functions consist of unconnected points. The gaps in the domain of a sequence are the intervals between the positive integers. The graph of gold prices on page 5 and the graph in Example 1 of this lesson are both examples of graphs of discrete functions.

Questions

COVERING THE IDEAS

- Consider the increasing sequence 1, 3, 5, 7, ... of positive odd numbers.
 - 13 is the 7th ? of the sequence.
 - If this sequence is called D , what is D_{11} ?
- Consider the equation $a_{11} = 22.83$.
 - Which number is the subscript?
 - What does the *number that is not the subscript* represent?
 - Which term of the sequence is this?
 - Rewrite the equation using function notation.
 - Rewrite the equation in words.

In 3 and 4, consider the sequence T of triangular numbers in the Activity on page 53.

- Compute the 20th triangular number.
- If $T_n = 15$, what is the value of n ?
- Draw a possible next term in the sequence at the right.
 - How many dots does it take to draw each of the first 5 terms?
 - Determine an explicit formula for the sequence S_n if S_n = the number of dots in the n th term.
- Consider the sequence h whose first six terms are 231, 120, 91, 66, 45, 28.
 - What number is the 4th term?
 - How is the sentence " $h_5 = 45$ " read?
 - $h_6 = \underline{\quad?}$



In 7 and 8, an explicit formula for a sequence is given. Write the first four terms of the sequence.

- $a_n = 7.3 - 3n$
- $S_n = \frac{n(n+1)(2n+1)}{6}$ (sum of the first n squares)

9. Refer to Solution 2 of Example 2.
- Graph the values from the table with the anniversary n on the horizontal axis and the total savings S_n on the vertical axis.
 - Should the points on the graph be connected? Why or why not? Relate your answer to the domain of the function.
10. a. **Multiple Choice** Which could be a formula for the n th term of the sequence 3, 9, 27, 81, ... ?
- A $t_n = 3n$ B $t_n = 3n^3$ C $t_n = 3^n$
- For the choices not used, write the first four terms of the sequence being represented.

APPLYING THE MATHEMATICS

11. a. Generate a table of the 4th through 7th terms of the sequence C defined by $C_n = \frac{n^2(n+1)^2}{4}$.
- This sequence gives the sum of the first n cubes. What does C_7 represent?

In Questions 12 and 13, consider the story *Anno's Magic Seeds* by Mitsumasa Anno (1992). It is the story of Jack, who receives two magic seeds. A person who eats one of these seeds will be full for one year, and planting a seed yields two seeds.

12. Jack begins the first year with two magic seeds. He eats one and plants the other. It grows and produces two seeds at the start of the second year. In this second year, Jack repeats his behavior by eating one seed and planting the other to get two seeds for the start of the third year. Jack continues the trend each year, eating one seed and planting the remaining seed to get two for the following year.
- Create a table and a graph illustrating the relationship between the year and the number of seeds Jack has at the beginning of the year for the first five years.
 - Does this situation determine a function? Why or why not?
 - Does this situation determine a sequence? Why or why not?
13. Suppose Jack decides to forgo eating a seed in the first year and instead plants both seeds to end up with four seeds at the start of the second year. In the second year, Jack eats one seed and plants the remaining three to end up with six seeds at the start of the third year. The third year, Jack eats one seed and plants the remaining five to end up with ten seeds at the start of the fourth year. Jack continues his behavior of eating one seed and planting what is left.



- a. Write the first six ordered pairs that relate the year to the number of seeds.
- b. Find an explicit formula for the number of seeds at the beginning of the n th year, for all $n > 1$.
14. Some common bacterial cells, such as *E. coli*, can divide and double every 20 minutes. The doubling process takes place when a microbe reproduces by splitting to make 2 cells. Each of these cells then splits in half to make a total of 4 cells. Each of these 4 cells then splits to make a total of 8, and so on. Each splitting is called a *generation*. If a colony begins with 125 microbes, the equation $P_n = 125(2)^{n-1}$ gives the number of microbes in the n th generation (assuming all microbes survive).
- a. Calculate the first three terms of the sequence.
- b. Identify the independent variable and dependent variable of the function.



E. coli cells

REVIEW

15. Solve for s in the formula $d = 7s - 13$. (Lesson 1-7)
16. The measure θ of an exterior angle of a regular polygon is given by $\theta = \frac{360}{n}$, where n is the number of sides of the polygon. Solve for n in terms of θ . (Lesson 1-7)

In 17 and 18, an equation is given. (Lesson 1-6)

- a. Solve the equation.
- b. Check your answer.

17. $5r - (2r + 1) = 6$

18. $7t - (9 - 4t) = 9$

19. A medium pizza costs \$12.50 plus \$1.50 for each topping. If C is the total cost of the pizza and t is the number of toppings ordered, then $C(t) = 12.50 + 1.50t$ gives c as a function of t . (Lesson 1-4)
- a. Specify the domain of this function.
- b. Write the four smallest numbers in the range of this function.
- c. Graph this function.
20. **Fill in the Blank** Let $h: a \rightarrow a^3 + 2$. Then $h: -3 \rightarrow \underline{\quad?}$. (Lesson 1-3)

EXPLORATION

21. Triangular numbers have many curious properties. For example, a triangular number can never end with the digits 2, 4, 7, or 9. Find one more property of the triangular numbers.

QUIZ YOURSELF ANSWERS

1. $T_4 = 10$; $T_5 = 15$
2. 120; 5050
3. $p(n) = \left(\frac{1}{2}\right)^n$