Lesson 8-9 Remembering Properties of Powers and Roots

BIG IDEA If you forget any of the properties of powers, you can recall them by testing special cases, following patterns, and knowing alternate ways of rewriting the same expression.

Mental Math

Evaluate.

a. $|-19 + -42|$

b. $|55.5 - 32| + |-9|$

c. $|-10 + 5| - |-30 + 15|$

Here are six properties of powers that were presented in this chapter. They apply to all exponents *m* and *n* and nonzero bases *a* and *b*.

Properties of Powers

Product of Powers
$b^m \cdot b^n = b^{m+n}$

Negative Exponent
$b^{-n} = \frac{1}{b^n}$

Power of a Power
$(b^m)^n = b^{mn}$

Quotient of Powers
$\frac{b^m}{b^n} = b^{m-n}$

Power of a Product
$(ab)^n = a^n b^n$

Power of a Quotient
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

In addition, two properties of square roots were studied. They apply to all nonnegative numbers *a* and *b*.

Properties of Square Roots

Product of Square Roots
$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$

Quotient of Square Roots
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, $b \neq 0$

With so many properties, some students confuse them. This lesson shows you how to use general problem-solving strategies to remember the properties and to test a special case to verify your reasoning.

Testing a Special Case

Because of calculators, a strategy called *testing a special case* is often possible. You can use this strategy to verify your reasoning.

Example 1

Emily was not sure how to simplify $x^5 \cdot x^6$. She felt the answer could be x^{30}, $2x^{11}$, or x^{11}. Which is correct?

Solution 1 Use a special case. Let $x = 3$. Now calculate $x^5 \cdot x^6$ (with a calculator) and see if it equals the calculator result for x^{30} or $2x^{11}$ or x^{11}.

$x^5 \cdot x^6 = 3^5 \cdot 3^6 = 177{,}147$

$x^{30} = 3^{30} = 205{,}891{,}132{,}094{,}649 \approx 2.0589 \cdot 10^{14}$

$2x^{11} = 2 \cdot 3^{11} = 354{,}294$

$x^{11} = 3^{11} = 177{,}147$

So, the answer is x^{11}.

Solution 2 Use repeated multiplication to rewrite x^5 and x^6.

$x^5 \cdot x^6 = (x \cdot x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x \cdot x)$

Notice there are 11 factors of x in the product. So, $x^5 \cdot x^6 = x^{11}$.

Showing That a Pattern Is Not Always True

When testing a special case, you should be careful in choosing numbers. The numbers 0, 1, and 2 are not good for checking answers to problems involving powers, because a pattern may work for a few of these numbers but not for all numbers. Recall that a *counterexample* is a special case for which the answer is false. To show a pattern is not true, it is sufficient to find one counterexample.

Example 2

Sir Lancelot's assistant, Squire Root, noticed $2^3 + 2^3 = 2^4$ since $8 + 8 = 16$. He guessed that in general, there is a property $x^3 + x^3 = x^4$.

He tested a second case by letting $x = 0$ and found $0^3 + 0^3 = 0^4$. He concluded that the property is always true. Is he correct?

Solution Try a different value for x. Let $x = 5$.

Does $5^3 + 5^3 = 5^4$? Does $125 + 125 = 625$? No. $x = 5$ is a counterexample that shows that Squire's property is not always true.

If you have trouble remembering a property or are not certain that you have simplified an expression correctly, try using repeated multiplication or testing a special case.

In Example 3, using the properties of powers, a calculation that is complicated even with a calculator, is reduced to two calculations.

Example 3

The mean radius of Earth is about $3.96 \cdot 10^3$ miles. The mean radius of Jupiter is about $4.34 \cdot 10^4$ miles. Using the formula $V = \frac{4}{3}\pi r^3$ for the volume V of a sphere with radius r, how many times could Earth fit inside Jupiter?

Solution To answer the question, you need to divide the volume of Jupiter by the volume of Earth. So substitute for r in the formula for the volume of a sphere.

$$\frac{\text{volume of Jupiter}}{\text{volume of Earth}} \approx \frac{\frac{4}{3}\pi(4.34 \cdot 10^4)^3}{\frac{4}{3}\pi(3.96 \cdot 10^3)^3}$$

$$= \frac{(4.34 \cdot 10^4)^3}{(3.96 \cdot 10^3)^3}$$ Multiplication of Fractions

$$= \left(\frac{4.34 \cdot 10^4}{3.96 \cdot 10^3}\right)^3$$ Power of a Quotient Property

$$= \left(\frac{43.4}{3.96}\right)^3$$ Quotient of Powers Property

$$\approx 1{,}316$$ Arithmetic

In Example 3, notice that by using the properties, we reduced a complicated calculation to two operations: division of 43.4 by 3.96 and then cubing of the quotient.

Earth could fit inside Jupiter about 1,300 times. (Jupiter is *very* big compared to Earth!)

It is important to realize that the properties of numbers and operations are consistent. If you apply them correctly, you can find many paths to a correct solution. The result you get using some properties will not disagree with the results another person gets by correctly using other properties.

GUIDED

Example 4

If $\left(\frac{9q^{-5}}{6q^{-3}}\right)^{-7} = aq^n$, what are the values of a and n?

Solution 1 This question requires that the expression on the left side be simplified into the form aq^n.

(*continued on next page*)

$$\left(\frac{9q^{-5}}{6q^{-3}}\right)^{-7} = \frac{9^{\underline{\ ?\ }}\ q^{\underline{\ ?\ }}}{6^{\underline{\ ?\ }}\ q^{\underline{\ ?\ }}}$$ Apply the Power of a Power Property to eliminate the parentheses.

$$= \frac{9^{\underline{\ ?\ }}}{6^{\underline{\ ?\ }}}\ q^{\underline{\ ?\ }}$$ Apply the Quotient of Powers Property.

$$= \frac{6^{\underline{\ ?\ }}}{9^{\underline{\ ?\ }}}\ q^{\underline{\ ?\ }}$$ Apply the Negative Exponent Property $\left(b^{-n} = \frac{1}{b^n}\right)$.

$$= \left(\frac{6}{9}\right)^{\underline{\ ?\ }} q^{\underline{\ ?\ }}$$ Use the Power of a Quotient Property.

$$= \left(\frac{2}{3}\right)^{\underline{\ ?\ }} q^{\underline{\ ?\ }}$$ Rewrite the fraction in lowest terms.

Thus, $a = \left(\frac{2}{3}\right)^{\underline{\ ?\ }}$ and $n = \underline{\ ?\ }$.

Solution 2 Work with a partner and follow these steps.

1. Apply the Quotient of Powers Property inside the parentheses.
2. Write the fraction in lowest terms.
3. Apply the Power of a Power Property.

You should obtain the same answer as in Solution 1.

Questions

COVERING THE IDEAS

1. **Multiple Choice** For all nonzero values of n, $\frac{n^{40}}{n^{10}} =$

   A n^4. B n^{30}. C 1^{30}. D 1^4.

2. **Multiple Choice** For all nonzero values of s, $\frac{s^4}{(2s)^2} =$

   A $4s^2$. B $\frac{s^2}{2}$. C $\frac{s^2}{4}$. D 1.

3. **Multiple Choice** For all nonzero values of v, $\frac{v^8 \cdot v^{12}}{(v^8)^{12}} =$

   A 1. B $-v^{76}$. C v^{-76}. D v^{76}.

4. **Multiple Choice** For all values of m and n, $(3m)^2 \cdot (2n)^3 =$

   A $6m^2n^3$. B $48m^2n^3$. C $72m^2n^3$. D $3{,}125m^2n^3$.

5. What is a counterexample?

6. **True or False** If two special cases of a pattern are true, then the pattern is true.

7. **True or False** If one special case of a pattern is false, then the pattern is false.

8. Consider the equation $x^4 = 8x$.
   a. Is the equation true for the special case $x = 2$?
   b. Is the equation true for the special case $x = 0$?
   c. Is the equation true for the special case $x = 3$?
   d. Is the equation true for all values of x?

In 9–11, test special cases to decide whether the pattern is always true. Show all work.

9. $(r^3)^{-4} = r^{3-4}$ 10. $(2n)^3 = 2n^3$ 11. $5y^3 \cdot 4y^4 = 20y^7$

In 12–15, name the property or properties being used.

12. $\left(\frac{2}{q}\right)^{10} = \frac{2^{10}}{q^{10}}$

13. $(x + 4)(x + 4)^4 = (x + 4)^5$

14. $(2x^2y)^3 = 2^3x^6y^3$

15. $\left(\frac{2}{41}\right)^{-3} = \left(\frac{1}{\frac{2}{41}}\right)^3$

In 16–18, write the expression without negative exponents.

16. $\left(\frac{3}{5}\right)^{-1}$ 17. $\left(\frac{3x^2}{y^3}\right)^{-2}$ 18. $\left(\frac{2m^{-2}}{12m}\right)^{-40}$

In 19 and 20, refer to Example 3 and use the fact that the mean radius of Mars is about $2.11 \cdot 10^3$ miles.

19. About how many times could Mars fit into Earth?

20. About how many times could Mars fit into Jupiter?

APPLYING THE MATHEMATICS

21. Describe two different ways to simplify $\left(\frac{x^6}{x^3}\right)^{-2}$.

22. Consider the pattern $\frac{2}{x} - \frac{1}{y} = \frac{2y - x}{xy}$.
   a. Is the pattern true when $x = 3$ and $y = 5$?
   b. Test the special case when $x = y$.
   c. Test another case. Let $x = 6$ and $y = 2$.
   d. Do you think the pattern is *always*, *sometimes but not always*, or *never true* for all nonzero x and y?

In 23 and 24, use the fact that the prime factorization of 24 is $2^3 \cdot 3$ and the prime factorization of 360 is $2^3 \cdot 3^2 \cdot 5$.

23. Give the prime factorization of 360^{25}.

24. Give the prime factorization of $\left(\frac{360}{24}\right)^8$.

REVIEW

25. Find the distance between the points (–3, –9) and (12, 56). Round to the nearest tenth, if necessary. **(Lesson 8-8)**

26. Let $y = \sqrt{3}$ and $z = \sqrt{5}$. Evaluate $\frac{y}{z} \cdot (2yz)^3$. **(Lesson 8-7)**

27. An isosceles right triangle is a right triangle in which both legs have the same length. Lenora says that an isosceles right triangle with legs of length s always has a hypotenuse of length $s \cdot \sqrt{2}$. Is Lenora correct? Why or why not? **(Lesson 8-6)**

28. Three days a week, Rollo rollerblades to work. He rollerblades eight and a half blocks. On each block, there are fourteen houses. Each house has three trees in its front yard. How many trees does Rollo rollerblade by on his way to work each week? **(Lesson 8-1)**

29. A line has an x-intercept of 12 and a y-intercept of 15. Write an equation of this line. **(Lesson 6-6)**

30. On a regular die, which is more likely: rolling an even number six times in a row, or rolling a number less than 3 four times in a row? **(Lesson 5-7)**

EXPLORATION

31. Six of the eight properties mentioned at the beginning of this lesson involve either multiplication or division. Show that *none* of these six properties is true if every multiplication is replaced by addition and every division is replaced by subtraction.

The first known roller skates were created in the 1760s and possessed a single line of wheels.

Source: National Museum of Roller Skating