Lesson 8-7 Multiplying and Dividing Square Roots

Vocabulary

radicand

BIG IDEA Like powers, square roots distribute over products and quotients.

Mental Math

Given $f(x) = 611x^2 + 492x - 1{,}000$. Calculate the following.

a. $f(0)$

b. $f(1)$

Activity 1

Step 1 Compute these square roots to the nearest thousandth either individually or using the list capability of a calculator.

$\sqrt{1} = 1.000$	$\sqrt{2} \approx 1.414$	$\sqrt{3} \approx 1.732$	$\sqrt{4} =$?
$\sqrt{5} \approx$?	$\sqrt{6} \approx 2.449$	$\sqrt{7} \approx$?	$\sqrt{8} \approx$?
$\sqrt{9} =$?	$\sqrt{10} \approx$?	$\sqrt{11} \approx$?	$\sqrt{12} \approx$?
$\sqrt{13} \approx$?	$\sqrt{14} \approx$?	$\sqrt{15} \approx$?	$\sqrt{16} =$?
$\sqrt{17} \approx$?	$\sqrt{18} \approx$?	$\sqrt{19} \approx$?	$\sqrt{20} \approx$?

Step 2 Consider the product $\sqrt{2} \cdot \sqrt{3}$. Find the product of the decimal approximations, rounded to 3 decimal places.

Decimal approximations: ? • ? ≈ 2.449

Is the decimal product found in the table above? ?

If so, write the equation that relates the product of the square roots.

Square roots: ? • ? = ?

Step 3 Repeat Step 2 but use a product of two different square roots from the list $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$.

Square roots: ? • ?

Decimal approximations: ? • ? ≈ ?

Is the decimal product found in the table above? ?

If so, write the equation that relates the product of the square roots.

Square roots: ? • ? = ?

Step 4 Multiply another pair of square roots in the table. ? • ?

Predict what their product will be. ? Is your prediction correct? ?

In Activity 1, you should have discovered that when the product of two numbers a and b is a third number c, it is also the case that the product of the square root of a and the square root of b is the square root of c. That is, if $ab = c$, then $\sqrt{a} \cdot \sqrt{b} = \sqrt{c} = \sqrt{ab}$. For example, because $5 \cdot 6 = 30$, $\sqrt{5} \cdot \sqrt{6} = \sqrt{30}$. You can check this by using decimal approximations to the square roots.

Product of Square Roots Property

For all nonnegative real numbers a and b, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

The Product of Square Roots Property may look unusual when the square roots are written in radical form. But when the square roots are written using the exponent $\frac{1}{2}$, the property takes on a familiar look.

$$a^{\frac{1}{2}} \cdot b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$$

It is just the Power of a Product Property, with $n = \frac{1}{2}$! This is further evidence of the appropriateness of thinking of the positive square root of a number as its $\frac{1}{2}$ power.

Activity 2

Step 1 Pick a square root from $\sqrt{6}$, $\sqrt{12}$, and $\sqrt{18}$.
Pick a square root from $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$.
Find the quotient of the decimal approximations.

Square roots __?__ ÷ __?__

Decimal approximations __?__ ÷ __?__ ≈ __?__

Is the quotient found in the table in Activity 1? __?__

If so, write the quotient as a square root. __?__

If not, is the quotient close to a number in the table? __?__

What square root is it closest to? __?__

Step 2 Repeat the process in Step 1 using a different square root from each group.

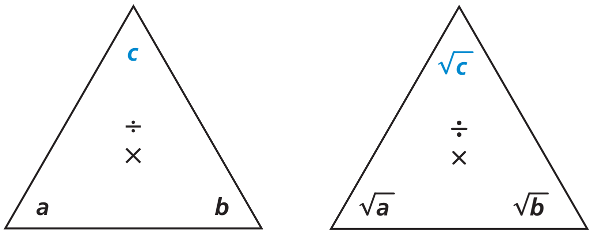
Step 3 Repeat the process again using a third pair of square roots.

In Activity 2, you should have discovered that when the quotient of two numbers c and a is a third number b, it is also the case that the quotient of the square root of c and the square root of a is the square root of b. That is, if $\frac{c}{a} = b$, then $\frac{\sqrt{c}}{\sqrt{a}} = \sqrt{\frac{c}{a}} = \sqrt{b}$. For example, since $\frac{24}{8} = 3$, $\frac{\sqrt{24}}{\sqrt{8}} = \sqrt{\frac{24}{8}} = \sqrt{3}$.

Quotient of Square Roots Property

For all positive real numbers a and c, $\frac{\sqrt{c}}{\sqrt{a}} = \sqrt{\frac{c}{a}}$.

Fact triangles can be used to visualize the Product of Square Roots Property and the Quotient of Square Roots Property. For all positive numbers *a, b,* and *c:*



$a \cdot b = c$
$b \cdot a = c$
$\frac{c}{a} = b$
$\frac{c}{b} = a$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{c}$
$\sqrt{b} \cdot \sqrt{a} = \sqrt{c}$
$\frac{\sqrt{c}}{\sqrt{a}} = \sqrt{b}$
$\frac{\sqrt{c}}{\sqrt{b}} = \sqrt{a}$

STOP **QY1**

QY1

Use either the Product or Quotient of Square Roots Property to evaluate each expression.

a. $\sqrt{8} \cdot \sqrt{2}$

b. $\frac{\sqrt{45}}{\sqrt{5}}$

c. $\frac{\sqrt{80}}{\sqrt{40}}$

"Simplifying" Radicals

A radical expression is said to be simplified if the quantity under the radical sign, called the **radicand,** has no perfect square factors other than 1.

Just as you can multiply square roots by using the Product of Square Roots Property, $\sqrt{4} \cdot \sqrt{10} = \sqrt{4 \cdot 10} = \sqrt{40}$, you can rewrite a square root as a product by factoring the radicand.

$$\begin{aligned}\sqrt{40} &= \sqrt{4 \cdot 10}\\ &= \sqrt{4} \cdot \sqrt{10}\\ &= 2 \cdot \sqrt{10}\end{aligned}$$

Many people consider $2\sqrt{10}$ to be simpler than $\sqrt{40}$ because it has a smaller radicand. This process is called *simplifying a radical*. The key to the process is to find a perfect square factor of the radicand.

Example 1

Simplify $\sqrt{27}$.

Solution Perfect squares larger than 1 are 4, 9, 16, 25, 36, 49,... . Of these, 9 is a factor of 27.

(*continued on next page*)

$\sqrt{27} = \sqrt{9 \cdot 3}$ Factor 27.

$= \sqrt{9} \cdot \sqrt{3}$ Product of Square Roots Property

$= 3\sqrt{3}$ $\sqrt{9} = 3$

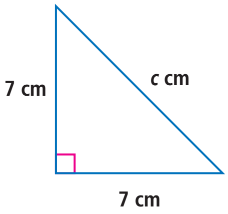
Check Using a calculator we see $\sqrt{27} \approx 5.196152423$ and $3\sqrt{3} \approx 5.196152423$.

Is $3\sqrt{3}$ really simpler than $\sqrt{27}$? It depends. For estimating purposes, $\sqrt{27}$ is simpler since we can easily see it is slightly larger than $\sqrt{25}$ or 5. But for seeing patterns, $3\sqrt{3}$ may be simpler. In the next example, the answer $7\sqrt{2}$ is related to the given information in a useful way that is not served by leaving it in the unsimplified form $\sqrt{98}$.

GUIDED

Example 2

Each leg of the right triangle below is 7 cm long.



a. Find the exact length of the hypotenuse.

b. Put the exact length in simplified radical form.

Solutions

a. Use the Pythagorean Theorem.

$c^2 = \underline{\ ?\ }^2 + \underline{\ ?\ }^2$ Substitute the lengths of the legs.

$c^2 = 98$ Add.

$c = \underline{\ ?\ }$ Use a radical sign to write the exact answer.

b. Now use the Product of Square Roots Property to simplify the result. Note that the perfect square 49 is a factor of 98.

$c = \sqrt{\underline{\ ?\ } \cdot 2}$

$c = \sqrt{\underline{\ ?\ }} \cdot \sqrt{\underline{\ ?\ }}$

$c = \underline{\ ?\ }\sqrt{2}$

The exact length of the hypotenuse is $\sqrt{98}$ or $\underline{\ ?\ }$ cm.

The Product of Square Roots Property also applies to expressions containing variables.

GUIDED

Example 3

Assume x and y are positive. Simplify $\sqrt{48x^2y^2}$.

Solution

$$\sqrt{48x^2y^2} = \sqrt{\underline{\ ?\ }} \cdot \sqrt{3} \cdot \sqrt{x^2} \cdot \sqrt{y^2}$$
$$= \underline{\ ?\ } \cdot \sqrt{3} \cdot x \cdot y$$
$$= \underline{\ ?\ }\, xy\sqrt{3}$$

Check Substitute values for x and y. We choose $x = 4$ and $y = 3$.

$$\sqrt{48x^2y^2} = \sqrt{48 \cdot 16 \cdot 9} \qquad 4xy\sqrt{3} = 4 \cdot 4 \cdot 3\sqrt{3}$$
$$= \sqrt{\underline{\ ?\ }} \qquad = \underline{\ ?\ }\sqrt{3}$$
$$\approx 83.14 \qquad \approx 83.14$$

It checks.

STOP QY2

Although square roots were first used in connection with geometry, they also have important applications in physical situations. One such application is with the pendulum clock.

In a pendulum clock, a clock hand moves each time the pendulum swings back and forth. The first idea for a pendulum clock came from the great Italian scientist Galileo Galilei in 1581. (At the time of Galileo, there was no accurate way to tell time; watches and clocks did not exist. People used sand timers but they were not very accurate.) Galileo died in 1642, before he could carry out his design. The brilliant Dutch scientist Christiaan Huygens applied Galileo's concept of tracking time with a pendulum swing in 1656.

A very important part of constructing the clock was calculating the time it takes a pendulum to complete one swing back and forth. This is called the *period* of the pendulum. The formula $p = 2\pi\sqrt{\frac{L}{32}}$ gives the time p in seconds for one period in terms of the length L (in feet) of the pendulum.

Example 4

A pendulum clock makes one "tick" for each complete swing of the pendulum. If a pendulum is 2 feet long, how many ticks would the clock make in one minute?

Solution First calculate p when $L = 2$.

(*continued on next page*)

QY2

a. Assume x and y are positive. Simplify $\sqrt{25x^2y}$.

b. Assume x is positive. Simplify $\frac{\sqrt{24x^2}}{\sqrt{6x^2}}$.

Dutch mathematician, Christiaan Huygens (1629–1695), patented the first pendulum clock, which greatly increased the accuracy of time measurement.

Source: University of St. Andrews

$p = 2\pi\sqrt{\frac{2}{32}} = 2\pi\sqrt{\frac{1}{16}} = 2\pi\frac{\sqrt{1}}{\sqrt{16}} = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$

It takes $\frac{\pi}{2}$ seconds for the pendulum to go back and forth.

$\frac{1 \text{ tick}}{\frac{\pi}{2} \text{ s}} \cdot \frac{60 \text{ s}}{1 \text{ min}} = \frac{60}{\frac{\pi}{2}} \frac{\text{tick}}{\text{min}} \approx 38.2 \text{ ticks/min}$

So the clock makes about 38.2 ticks per minute.

Questions

COVERING THE IDEAS

In 1–4, use the Product or Quotient of Square Roots Property to evaluate the expression.

1. $\sqrt{8} \cdot \sqrt{2}$

2. $\sqrt{36 \cdot 81 \cdot 100}$

3. $\frac{\sqrt{40}}{\sqrt{10}}$

4. $\frac{\sqrt{6^3}}{\sqrt{6}}$

5. If $\sqrt{3} \cdot \sqrt{6} = \sqrt{x} = y\sqrt{z}$, what is x, what is y, and what is z?

6. If $\frac{\sqrt{63}}{\sqrt{7}} = \sqrt{x} = y$, what is x and what is y?

7. Multiple Choice Which is *not* equal to $\sqrt{50}$?

A $\sqrt{5} \cdot \sqrt{10}$

B $\sqrt{25} + \sqrt{25}$

C $\sqrt{2} \cdot \sqrt{25}$

D $5\sqrt{2}$

8. a. Use the formula $p = 2\pi\sqrt{\frac{L}{32}}$ to calculate the time p for one period of a pendulum of length $L = 8$ feet.

b. If the clock makes one tick for each pendulum swing back and forth, how many ticks are there in one minute?

In 9–12, simplify the square root.

9. $\sqrt{18}$

10. $\sqrt{24}$

11. $\sqrt{50}$

12. $8\sqrt{90}$

13. Assume m and n are positive. Simplify each expression.

a. $\sqrt{150m^2n}$

b. $\frac{\sqrt{112m^7}}{\sqrt{7m^3}}$

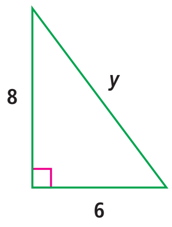
14. The length of each leg of a right triangle is 8 cm. What is the exact length of the hypotenuse?

15. Let $m = \frac{1}{2}$ in the Power of a Quotient Property $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$. What property of this lesson is the result?

APPLYING THE MATHEMATICS

In 16–19, write the exact value of the unknown in simplified form. Then approximate the unknown to the nearest hundredth.

16.



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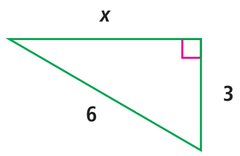

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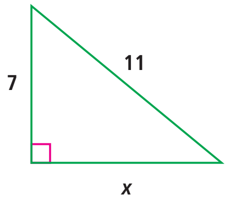

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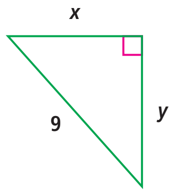


20. Find the area of a triangle with base $\sqrt{18}$ and height $6\sqrt{2}$.

21. The radical $\sqrt{50}$ is equivalent to $5\sqrt{2}$. Explain why it is easier to tell that $\sqrt{50}$ is slightly larger than 7 than it is to tell $5\sqrt{2}$ is slightly larger than 7.

In 22–25, explore adding square roots. You can add square roots using the Distributive Property if their radicands are alike. So, $3\sqrt{11} + 5\sqrt{11} = 8\sqrt{11}$, but $2\sqrt{11} + 4\sqrt{3}$ cannot be simplified. In each expression below, simplify terms if possible, then add or subtract.

22. $2\sqrt{25} + \sqrt{49}$

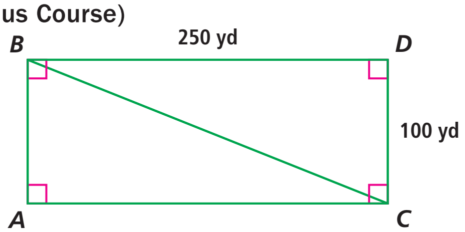
23. $\sqrt{12} - 10\sqrt{3}$

24. $\sqrt{45} - \sqrt{20}$

25. $4\sqrt{50} + 3\sqrt{18}$

REVIEW

In 26 and 27, consider the rectangular field pictured here. (Lesson 8-6, Previous Course)



26. How much shorter would it be to walk diagonally across the field as opposed to walking along the sides to get from B to C?

27. Suppose A is the origin of the coordinate system with the y-axis on $\overleftrightarrow{AB}$ and the x-axis on $\overleftrightarrow{AC}$. Give the coordinates of point D.

In 28–30, write the expression as a power of a single number. (Lessons 8-4, 8-3, 8-2)

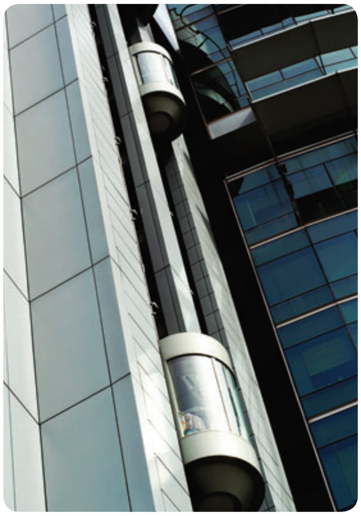
28. $\frac{k^{15}}{k^9}$ **29.** $x^4 \cdot x$ **30.** $(w^2)^{-3}$

31. Which is greater, $(6^4)^2$, or $6^4 \cdot 6^2$? **(Lesson 8-2)**

32. In 1995, Ellis invested \$5,000 for 10 years at an annual yield of 8%. In 2005, Mercedes invested \$7,000 for 5 years at 6%. By the end of 2010, who would have more money? Justify your answer. **(Lesson 7-1)**

33. After x seconds, an elevator is on floor y, where $y = 46 - 1.5x$. Give the slope and y-intercept of $y = 46 - 1.5x$, and describe what they mean in this situation. **(Lesson 6-4)**

34. A box with dimensions 30 cm by 60 cm by 90 cm will hold how many times as much as one with dimensions 10 cm by 20 cm by 30 cm? **(Lesson 5-10)**



Elisha Graves Otis invented the first safety brake for elevators in 1852, kick-starting the elevator industry.

Source: Elevator World, Inc.

EXPLORATION

35. Is there a Product of Cube Roots Property like the Product of Square Roots Property? Explore this idea and reach a conclusion. Describe your exploration and defend your conclusion.

36. Use the formula $p = 2\pi\sqrt{\frac{L}{32}}$ to determine the length of a pendulum that will make 1 tick each second. Answer to the nearest hundredth of an inch.

QY ANSWERS

1a. 4

1b. 3

1c. $\sqrt{2}$

2a. $5x\sqrt{y}$

2b. 2