Lesson 8-6

Square Roots and Cube Roots

Vocabulary

square
squared
square root
radical sign ($\sqrt{}$)
cube
cubed
cube root

▶ BIG IDEA If a first number is the square (or the cube) of a second number, then the second number is a square root (or cube root) of the first.

Mental Math

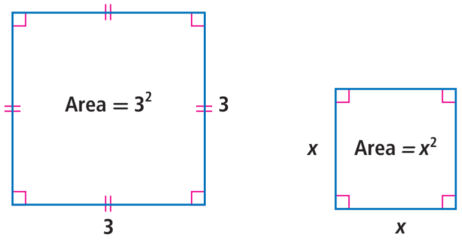
In each set, which does not equal the others?

a. $\frac{3x}{5}$, $\frac{3}{5}x$, $\frac{3}{5x}$

b. $\frac{y}{9}$, $y \cdot \frac{1}{9}$, $\frac{1}{9} \cdot \frac{1}{y}$, $y \div 9$

Areas of Squares and Powers as Squares

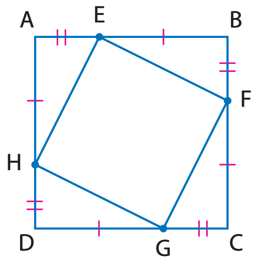
The second power x^2 of a number x is called the **square** of x, or x **squared,** because it is the area of a square with side length x. This is not a coincidence. The ancient Greek mathematicians pictured numbers as lengths, and they pictured the square of a number as the area of a square.



It is easy to calculate the area of a square on a grid if the square's sides are horizontal and vertical, but what if the square's sides are slanted?

Activity 1

Follow these steps to determine the area of the square *EFGH*, at the right.



Step 1 Square *ABCD* is 3 units on a side. What is its area?

Step 2 Triangle *AEH* is a right triangle with legs of length 1 unit and 2 units. What is the area of $\triangle AEH$?

Step 3 Subtract the areas of the four corner triangles from the area of *ABCD* to get the area of *EFGH*.

Sides of Squares and Square Roots

You should have found that the area of *EFGH* is 5 square units. If the area of the square *EFGH* is 5 square units, what is the length of one of its sides? The Greek mathematicians could do the previous calculations easily. But now they were stumped. Can *GH* be $2\frac{1}{2}$? No, because $\left(2\frac{1}{2}\right)^2 = 2.5^2 = 6.25$, which is greater than 5. In fact, the Greeks were able to show that it is impossible to find any simple fraction whose square is exactly 5. So they simply called the length the *square root* of 5. We still do that today. The length of a side of a square whose area is x is called a square root of x. The length of *GH* is a square root of 5. Similarly, a square root of 9 is 3, because a square with area 9 has side 3.

Definition of Square Root

If $A = s^2$, then s is a **square root** of A.

If two numbers have the same absolute value, such as 3 and –3, then they have the same square, 9. Although –3 cannot be the length of a side of a square, every positive number but 0 has two square roots, one positive and one negative. We denote the square roots of A by the symbols $\sqrt{A}$ (the positive root) and $-\sqrt{A}$ (the negative root). So the square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$. The two square roots of 5 are $\sqrt{5}$ and $-\sqrt{5}$. In the figure on the previous page, $GH = \sqrt{5}$.

The Radical Sign $\sqrt{}$

The **radical sign** $\sqrt{}$ indicates that a square root is being found. The horizontal bar attached to it, called a *vinculum*, acts like parentheses. The order of operations applies, so work is done inside the radical sign before the square root is taken. For example, $\sqrt{16 - 9} = \sqrt{7}$. On the other hand, $\sqrt{16} - \sqrt{9} = 4 - 3 = 1$.

In dealing with square roots, it helps to know the squares of small positive integers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...

Example 1

What are the square roots of each number?

a. 64

b. 17.3

(continued on next page)

Solutions

a. Because $8^2 = 64$ and $(-8)^2 = 64$, the square roots of 64 are 8 and -8. We can write $\sqrt{64} = 8$ and $-\sqrt{64} = -8$.

b. Because there is no decimal that multiplied by itself equals 17.3, just write $\sqrt{17.3}$ and $-\sqrt{17.3}$. A calculator shows $\sqrt{17.3} \approx 4.1593$ and so $-\sqrt{17.3} \approx -4.1593$.

Square Roots That Are Not Whole Numbers

The Greek mathematician Pythagoras and his followers, the Pythagoreans, were able to prove that numbers like $\sqrt{5}$ are not equal to simple fractions or ratios. Today we know that there is no finite or repeating decimal that equals $\sqrt{5}$. While $\sqrt{5}$ is approximately 2.23606797..., the decimal does not end nor repeat. You should check that the squares of truncated forms of 2.23606797..., are very close to 5. For example, $2.236 \cdot 2.236 = 4.999696$. But only $\sqrt{5}$ and $-\sqrt{5}$ square to be exactly 5, so $\sqrt{5} \cdot \sqrt{5} = 5 = -\sqrt{5} \cdot -\sqrt{5}$.

Square of the Square Root Property

For any nonnegative number x, $\sqrt{x} \cdot \sqrt{x} = \sqrt{x^2} = x$.

You can use this property to simplify or evaluate expressions that are exact, rather than use your calculator to deal with approximations.

STOP QY1

QY1

Explain why $4\sqrt{10} \cdot 3\sqrt{10} = 120$.

A Positive Square Root of *x* Is a Power of *x*

Suppose $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in the Product of Powers Property $x^m \cdot x^n = x^{m+n}$. Then, $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$.

This means that $x^{\frac{1}{2}}$ is a number which, when multiplied by itself, equals x. Thus $x^{\frac{1}{2}}$ is a square root of x, and we identify $x^{\frac{1}{2}}$ as the positive square root of x. So, for any positive number x, $x^{\frac{1}{2}} = \sqrt{x}$.

For example, $100^{\frac{1}{2}} = \sqrt{100} = 10$ and $64.289^{\frac{1}{2}} = \sqrt{64.289} \approx 8.02$.

Activity 2

Use a calculator to verify that $x^{\frac{1}{2}} = \sqrt{x}$.

Step 1a. Enter 16^(1/2). What number results?

b. You have calculated the square root of what number?

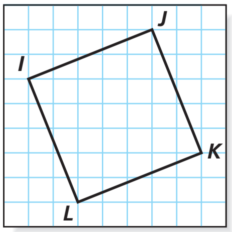
Step 2a. Enter 8^0.5. What number results?

b. You have calculated the square root of what number?

Step 3 Enter (−4)^(1/2). What results, and why?

Activity 3

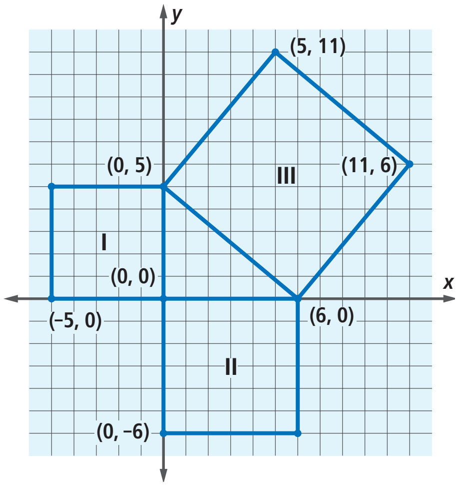
Use the idea of Activity 1 to determine the length of a side of square *IJKL* shown below. Show your work.



Activity 4

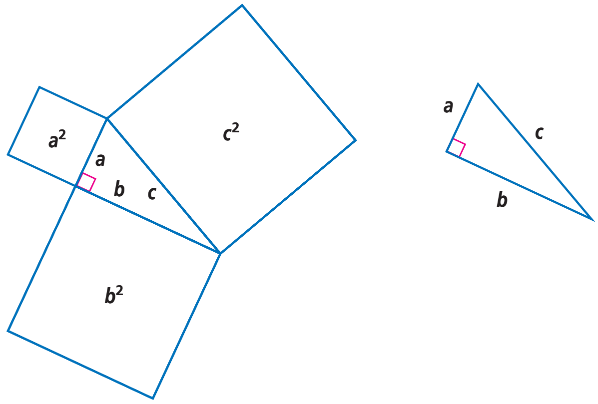
Three squares are drawn on a coordinate grid at the right.

1. Use the idea of Activities 1 and 3 to determine the area of square III. Explain your work.
2. What is the area of square I?
3. What is the area of square II?
4. How are the areas of the three squares related to each other?



The Pythagorean Theorem

The result of Activity 4 is one example of the *Pythagorean Theorem*. We state this theorem in terms of area first, and then in terms of powers.



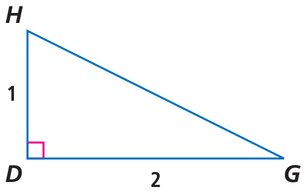
Pythagorean Theorem

(*In terms of area*) In any right triangle, the sum of the areas of the squares on its legs equals the area of the square on its hypotenuse.

(*In terms of length*) In any right triangle with legs of lengths a and b and a hypotenuse of length c, $a^2 + b^2 = c^2$.

For example, in $\triangle GDH$ from Activity 1, $HD^2 + DG^2 = GH^2$.

$$1^2 + 2^2 = GH^2$$
$$1 + 4 = GH^2$$
$$5 = GH^2$$

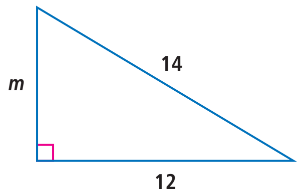


By the definition of square root, $GH = \sqrt{5}$.

The Pythagorean Theorem is perhaps the most famous theorem in all of mathematics. It seems to have been discovered independently in many cultures, for it was known to the Babylonians, Indians, Chinese, and Greeks well over 2,500 years ago. In the United States and Europe, this theorem is known as the Pythagorean Theorem because Pythagoras or one of his students proved it in the 6th century BCE. In China, it is called the Gougu Theorem. In Japan, it is called "The Theorem of the Three Squares."

Example 2

Use the Pythagorean Theorem to find the length of the missing side.



Solution Use the Pythagorean Theorem to write an equation involving the lengths of the three sides of the right triangle.

$$m^2 + 12^2 = 14^2$$
$$m^2 + 144 = 196$$
$$m^2 + 144 - 144 = 196 - 144$$
$$m^2 = 52$$
$$m = \sqrt{52}$$
$$m \approx 7.21$$

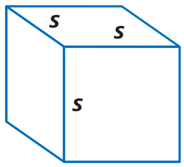
Check Substitute the solution into the original triangle and apply the Pythagorean Theorem.

Does $\left(\sqrt{52}\right)^2 + 12^2 = 14^2$?

$52 + 144 = 196$ Yes, it checks.

Cubes and Cube Roots

The third power x^3 of a number x is called the **cube** of x, or x **cubed,** because it is the volume of a cube with edge x. So, for example, the volume of a cube with edge of length 6 inches is $6 \cdot 6 \cdot 6$, or 216 cubic inches. We write $6^3 = 216$, and we say "6 cubed equals 216." Like the square, this is not a coincidence. The ancient Greek mathematicians pictured the cube of a number s as the volume of a cube whose edge is s.



Also, in a manner like that of a square, if the volume of a cube is V, then an edge of the cube is called a **cube root** of V.

Definition of Cube Root

If $V = s^3$, then s is a cube root of V.

Since $6^3 = 216$, 6 is a cube root of 216. Unlike square roots, cube roots do not come in pairs. For example, –6 is not a cube root of 216, since $(-6)^3 = -216$. In the real numbers, all numbers have exactly one cube root.

STOP **QY2**

The cube root of V is written using a radical sign as $\sqrt[3]{V}$. For example, $\sqrt[3]{216} = 6$ and $\sqrt[3]{-216} = -6$. Many calculators have a $\sqrt[3]{}$ command, though it may be hidden in a menu. You should try to locate this command on your calculator. However, you will learn an alternate method for calculating cube roots in the next lesson.

Cube of the Cube Root Property

For any nonnegative number x, $\sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} = \sqrt[3]{x^3} = x$.

For example, $1.2^3 = 1.2 \cdot 1.2 \cdot 1.2 = 1.728$. This means:

- 1.728 is the cube of 1.2.
- 1.2 is the cube root of 1.728.
- $1.2 = \sqrt[3]{1.728}$

When the value of a square root or cube root is not an integer, your teacher may expect two versions: (1) the exact answer written with a radical sign and (2) a decimal approximation rounded to a certain number of decimal places.

▶ **QY2**

Fill in the Blanks

Since $4^3 = 64$, __?__ is the cube root of __?__.

Questions

COVERING THE IDEAS

1. **a.** A side of a square is 16 units. What is its area?
   b. The area of a square is 16 square units. What is the length of a side?

2. Rewrite the following sentences, substituting numbers for x and y to produce a true statement. *A square has a side of length x and an area y.* Then y is the square of x, and x is the square root of y.

In 3–6, write or approximate the number to two decimal places.

3. $\sqrt{36}$

4. $\sqrt{121}$

5. $50^{\frac{1}{2}}$

6. $10^{0.5}$

In 7–10, evaluate the expression to the nearest thousandth.

7. $\sqrt{1{,}000}$

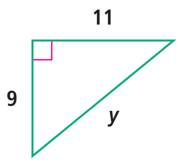
8. $\sqrt{100 + 100}$

9. $\sqrt{5} \cdot \sqrt{5}$

10. $2 \cdot \left(\frac{3}{4}\right)^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{2}}$

11. **a.** Approximate $\sqrt{11}$ to the nearest hundred-thousandth.
    b. Multiply your answer to Part a by itself.
    c. What property is validated by Parts a and b?

In 12–14, find the length of the missing side of the right triangle. If the answer is not an integer, give both its exact value and an approximation to the nearest hundredth.

12.


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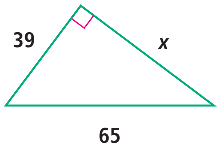

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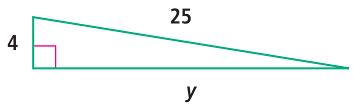


15. Write the cubes of the integers from 1 to 10.

16. 2 is a cube root of 8 because ___?___.

17. **a.** Write the exact cube root of 1,700.
    b. Estimate the cube root of 1,700 to the nearest thousandth.
    c. Check your answer to Part b by multiplying your estimate by itself three times.

In 18 and 19, evaluate the expression.

18. $\sqrt[3]{2.197}$

19. $\sqrt[3]{45} \cdot \sqrt[3]{45} \cdot \sqrt[3]{45}$

APPLYING THE MATHEMATICS

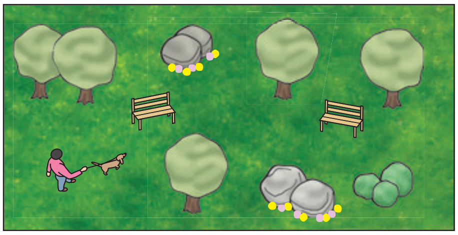
20. Suppose p is a positive number.

a. What is the sum of the square roots of p?

b. What is the product of the square roots of p?

21. In Chapter 7, the equation $P = 100{,}000(1.02)^x$ gave the population x years from now of a town of 100,000 today with a growth rate of 2% per year. Calculate P when $x = \frac{1}{2}$, and tell what the answer means.

22. A small park is shown below. If you want to go from one corner to the other corner, how many fewer feet will you walk if you go diagonally through the park rather than walk around it? Round your answer to the nearest foot.



23. In the movie *The Wizard of Oz,* the scarecrow recites the following after receiving his diploma, "The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side." The scarecrow was attempting to recite the Pythagorean Theorem.

a. Write several sentences explaining how this statement differs from the Pythagorean Theorem.

b. Is the scarecrow's statement accurate? If not, produce a counterexample.

24. A dog is on a leash that is 10 meters long and attached to a pole 2.5 meters above the the dog's collar. To the nearest tenth of a meter, how far from the pole can the dog roam?

REVIEW

25. As you know, $4 \cdot 9 = 36$. So the square of 2 times the square of 3 equals the square of 6. Determine the general pattern. **(Lesson 8-5)**

26. Simplify $a^6 \cdot \left(\frac{3}{a}\right)^3$. **(Lesson 8-5)**

In 27 and 28, solve. (Lessons 8-4, 8-2)

27. $3^4 \cdot 3^x = 3^{12}$

28. $\frac{1}{512} = 2^a$

29. Other than the sun, the star nearest to us, Proxima Centauri, is about $4 \cdot 10^{13}$ km away. Earth's moon is about $3.8 \cdot 10^5$ km from us. If it took astronauts about 3 days to get to the moon in 1969, at that speed how long would it take them to get to Proxima Centauri? **(Lesson 8-3)**

30. ***Skill Sequence*** Solve each equation for y. Assume $a \neq 0$. **(Lesson 4-7)**

a. $3x + 4y = 2$

b. $6x + 8y = 4$

c. $9x + 12y = 6$

d. $3ax + 4ay = 2a$

You would have to circumnavigate Earth $9\frac{1}{2}$ times to equal the distance from Earth to the moon.

EXPLORATION

31. Make a table to evaluate $n^{\frac{1}{3}}$ on your calculator when n is 1, 2, 3, ..., up to 7. What do you think $n^{\frac{1}{3}}$ is equivalent to? Give a reason for your answer.

QY ANSWERS

1. $4\sqrt{10} \cdot 3\sqrt{10}$
$= 4 \cdot 3 \cdot \sqrt{10} \cdot \sqrt{10}$
$= 12 \cdot 10$
$= 120$

2. 4; 64