Lesson

8-5 Powers of Products and Quotients

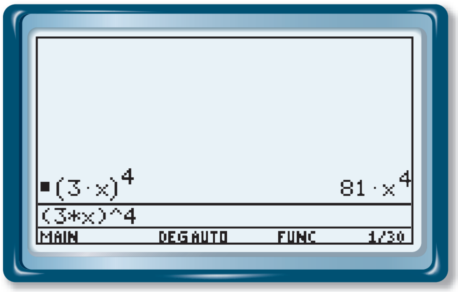
BIG IDEA Because of the relationship among multiplication, division, and powers, powers distribute over products and quotients.

The Power of a Product

The expression $(3x)^4$ is an example of a power of a product. It can be rewritten using repeated multiplication.

$(3x)^4 = (3x) \cdot (3x) \cdot (3x) \cdot (3x)$	Repeated Multiplication Model for Powering
$= 3 \cdot 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$	Associative and Commutative Properties
$= 3^4 \cdot x^4$	Repeated Multiplication Model for Powering
$= 81x^4$	Arithmetic

You can check this answer using a CAS.



In general, any positive integer power of a product can be rewritten using repeated multiplication.

$$(ab)^n = \underbrace{(ab) \cdot (ab) \cdot \ldots \cdot (ab)}_{n \text{ factors}}$$

$$= \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \cdot \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}$$

$$= a^n \cdot b^n$$

When a and b are nonzero, this result holds for all values of n.

Power of a Product Property

For all nonzero a and b, and for all n, $(ab)^n = a^n b^n$.

Mental Math

Find the slope of the line through

a. (–4.5, 19) and (90, 19).

b. (4, –1.5), (–3.5, –1.5).

c. (0, 0) and $\left(\frac{3}{4}, \frac{7}{4}\right)$.

This property can be applied to simplify the expression $(3x)^4$ from page 481. The power is applied to each factor of $3x$, so $(3x)^4 = 3^4x^4$, resulting in $81x^4$.

Example 1

Simplify $(-4x)^3$.

Solution Use the Power of a Product Property.

$(-4x)^3 = (-4)^3 \cdot x^3 = -64x^3$

Check Substitute a test value for x and follow order of operations.

Let $x = 1.5$. Does $(-4x)^3 = -64x^3$?

$$\begin{aligned} (-4 \cdot 1.5)^3 &= -64(1.5)^3 \\ (-6)^3 &= -64 \cdot (3.375) \\ -6 \cdot -6 \cdot -6 &= -216 \\ -216 &= -216 \end{aligned}$$

It checks.

Remember that in the order of operations, powers take precedence over opposites. In $-64x^3$, the power is done before the multiplication. In $(-4x)^3$, the multiplication is inside parentheses so it is done before the power.

STOP QY1

> **QY1**
> Simplify $(3xy)^4$.

GUIDED

Example 2

Simplify $(-5x^2y^3z)^3$.

Solution

$(-5x^2y^3z)^3$

$= (-5)^{\underline{?}}(x^2)^{\underline{?}}(y^3)^{\underline{?}}z^{\underline{?}}$ Apply the Power of a Product Property.

$= (-5)^{\underline{?}}x^{\underline{?}}y^{\underline{?}}z^{\underline{?}}$ Apply the Power of a Power Property.

$= \underline{\ ?\ }$ Evaluate the numerical power.

The Power of a Quotient

The expression $\left(\frac{a}{b}\right)^n$ is the power of a quotient. By using the properties of the previous lessons, you can write this without parentheses.

$$\left(\frac{a}{b}\right)^n = \left(a \cdot \frac{1}{b}\right)^n = (a \cdot b^{-1})^n = a^n \cdot (b^{-1})^n = a^n \cdot b^{-n} = \frac{a^n}{b^n}$$

Power of a Quotient Property

For all nonzero a and b, and for all n, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

The Power of a Quotient Property enables you to find powers of fractions more quickly.

Example 3

Write $\left(\frac{3}{4}\right)^5$ as a simple fraction.

Solution 1 Use the Power of a Quotient Property.

$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5} = \frac{243}{1{,}024}$

Solution 2 Use repeated multiplication.

$\left(\frac{3}{4}\right)^5 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{3^5}{4^5} = \frac{243}{1{,}024}$

Check Change the fractions to decimals.

$\left(\frac{3}{4}\right)^5 = 0.75^5 = 0.2373046875$

$\frac{243}{1{,}024} = 0.2373046875$

They are equal.

STOP **QY2**

Powers are found in many formulas for area and volume.

QY2

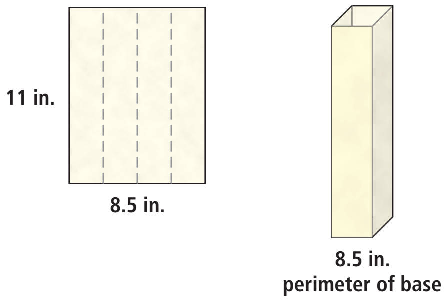
Rewrite $11 \cdot \left(\frac{2}{m}\right)^6$ as a simple fraction.

Activity

You will need two pieces of 8.5-in. by 11-in. paper, tape, scissors, and a ruler.

Step 1

Begin with one sheet of paper, positioned so that it is taller than it is wide. Fold it into fourths lengthwise and tape the long edges together to form the sides of a tall box with a square base.



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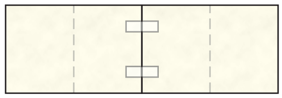
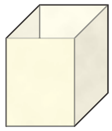
Step 2 Cut the other piece of paper in half to create two 8.5-in. by 5.5-in. pieces. Fold each half, as shown by the dotted lines. Tape these pieces together to form a 17-in. by 5.5-in. piece of paper. Tape the short edges together to form the sides of a short box with a square base.





Step 3 **Multiple Choice** Which of the following do you think is true?

A The tall, skinny box has more volume.

B The short, wide box has more volume.

C Both boxes have the same volume.

In several sentences, justify your conjecture with a logical argument.

Step 4 Test your conjecture using the formula $V = s^2h$ for the volume V of a box with height h and a square base whose sides have length s.

a. Calculate the length of the sides of the base of the tall prism. $s =$ __?__

b. Calculate the volume of the tall prism. $V =$ __?__

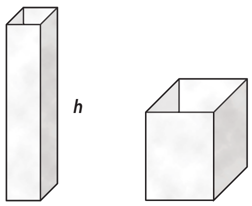
c. Repeat Parts a and b for the short prism. $s =$ __?__, $V =$ __?__

Step 5 According to your calculations, which is the correct answer to Step 3? __?__

Do you think you would get the same result if you started with a sheet of paper of a different size? Why or why not?

Using Powers of Quotients to Explain the Activity Results

Suppose you begin with a sheet of paper with height h and width p. The shorter box has half the height of the taller box, but the perimeter of its base is twice as long. Each side of the base of the tall box has length $\frac{p}{4}$. Each side of the base of the short box has length $\frac{2p}{4}$. So for the short prism, $2p =$ perimeter and $\frac{h}{2} =$ height.



The volume of a box with a square base is given by the formula $V = s^2h$, where the height is h and the side of the base is s. So, the volume of the tall box $= \left(\frac{p}{4}\right)^2 \cdot h$, and the volume of the short box $= \left(\frac{2p}{4}\right)^2 \cdot \frac{h}{2}$. To compare these volumes, we use properties of powers to simplify the expressions.

Example 4

The tall box has volume $\left(\frac{p}{4}\right)^2 \cdot h$ and the short box has volume $\left(\frac{2p}{4}\right)^2 \cdot \frac{h}{2}$.

a. Show that the volume of the tall box is always less than or equal to the volume of the short box.

b. The volume of the short box is how many times the volume of the tall one?

Solution

a. First apply the Power of a Quotient Property to simplify each volume.

Tall Box

$$V = \left(\frac{p}{4}\right)^2 \cdot h = \frac{p^2}{4^2} \cdot h = \frac{p^2}{16} \cdot h = \frac{p^2h}{16}$$

Short Box

$$V = \left(\frac{2p}{4}\right)^2 \cdot \frac{h}{2} = \left(\frac{(2p)^2}{4^2}\right) \cdot \frac{h}{2} = \left(\frac{4p^2}{16}\right) \cdot \frac{h}{2} = \left(\frac{p^2}{4}\right) \cdot \frac{h}{2} = \frac{p^2h}{8}$$

$$\text{Volume of the tall box} = \frac{p^2h}{16} = \frac{1}{2} \cdot \frac{p^2h}{8} = \frac{1}{2} \cdot \text{volume of short box}$$

Because the volume of the tall box is half the volume of the short one, the volume of the tall box is less than the volume of the short box.

b. The volume of the short box is 2 times the volume of the tall box.

Questions

COVERING THE IDEAS

1. a. Rewrite $(6x)^3$ without parentheses.
   b. Check your answer by letting $x = 2$.

In 2–5, rewrite the expression without parentheses.

2. $(5t^2)^3$
3. $8(-7xy)^3$
4. $2(x^2y)^4$
5. $(-t)^{93}$

6. Aisha made a common error when she wrote $(3x)^4 = 12x^4$. Show her this is incorrect by substituting 2 in for x. Then, write a note to Aisha explaining what she did wrong.

In 7–9, write as a simple fraction.

7. $\left(\frac{2}{3}\right)^4$

8. $5\left(\frac{n^5}{10}\right)^3$

9. $\left(\frac{19}{2y}\right)^3$

10. What is the area of a square with perimeter p?

APPLYING THE MATHEMATICS

11. The area A of an isosceles right triangle with leg L can be found using the formula $A = \frac{1}{2}L^2$. If L is multiplied by 6, what happens to the area of the triangle?

12. Suppose you tape a 3-in. by 5-in. notecard to a pencil widthwise (as shown in Figure 1). Assume that the radius of the round pencil is $\frac{3}{16}$ in.

a. If you rotate the pencil, what shape is traced by point A? Find the area of the shape.

b. If you rotate the pencil, the entire notecard in Figure 1 traces a cylinder. Cylinders with height h and a base of radius r have volume $V = \pi r^2 h$. Calculate the volume of this region.

Figure 1

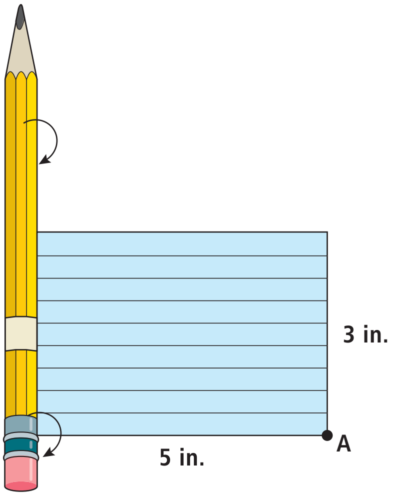
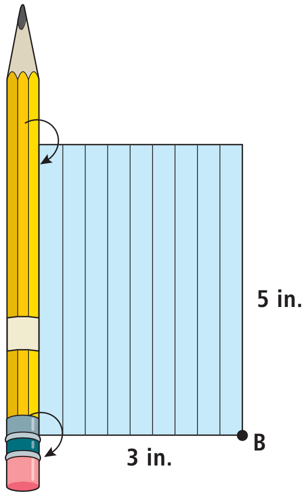


Figure 2



c. Suppose you tape a 3 in.-by-5 in. notecard to a pencil heightwise, as shown in Figure 2. If you rotate the pencil, what shape is traced by point B? Find the area of the shape.

d. If you rotate the pencil, what shape is traced by the entire notecard? Calculate the volume of this region.

e. True or False Changing the taping of the notecard does not change the volume of the shape that is traced by the notecard when the pencil is rotated.

In 13–15, rewrite without parentheses and simplify.

13. $(xy)^2\left(\frac{x}{y}\right)^3$

14. $(abc)^0 \cdot \frac{(ab)^2}{abc}$

15. $(2w)^4(3w^3)^2$

In 16–18, fill in the blank with an exponent or an expression that makes the statement true for all values of the variables.

16. $(3x^2y)^{\underline{?}} = 27x^6y^3$

17. $(2xy^2)^{\underline{?}} = 1$

18. $(\underline{\ ?\ })^3 = 64x^6y^9$

19. If $x = 5$, what is the value of $\frac{(3x)^9}{(3x)^7}$?

REVIEW

In 20 and 21, simplify the expression so that your answer does not contain parentheses or negative exponents. Then evaluate when $r = 1.5$ and $s = 1$. (Lessons 8-4, 8-3)

20. $r^4s^9r^{-3}s^7$

21. $\frac{17s^{-2}}{5^5} \cdot r^{-2}$

22. On each day (Monday through Friday) this week, Antoine will do one of three activities after school: play tennis, walk his dog, or read. How many different orders of activities are possible? **(Lesson 8-1)**

23. Solve $9(p - 2) < 47p - 2(5 - p)$ for p. **(Lesson 4-5)**

The World Junior Tennis competition, the international team competition for players aged 14 and under, was started by the International Tennis Federation in 1991.

Source: International Tennis Federation

EXPLORATION

24. A list of some powers of 3 is shown below. Look carefully at the last digit of each number.

$3^0 = 1$ $\quad 3^4 = 81$

$3^1 = 3$ $\quad 3^5 = 243$

$3^2 = 9$ $\quad 3^6 = 729$

$3^3 = 27$

a. Predict the last digit of 3^{10}. Check your answer with a calculator.

b. Predict the last digit of 3^{20}. Check your answer with a calculator.

c. Describe how you can find the last digit of any positive integer power of 3.

d. Does a similar pattern happen for powers of 4? Why or why not?

QY ANSWERS

1. $81x^4y^4$

2. $\frac{704}{m^6}$