Lesson

8-4 Negative Exponents

▶ **BIG IDEA** The numbers x^{-n} and x^n are reciprocals.

What Is the Value of a Power with a Negative Exponent?

You have used base 10 with a negative exponent to represent small numbers in scientific notation. For example, $10^{-1} = 0.1 = \frac{1}{10^1}$, $10^{-2} = 0.01 = \frac{1}{10^2}$, $10^{-3} = 0.001 = \frac{1}{10^3}$, and so on.

Now we consider other powers with negative exponents. That is, we want to know the meaning of b^n when n is negative. Consider this pattern of the powers of 2.

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$
$$2^0 = 1$$

Each exponent is one less than the one above it. The value of each power is half that of the number above. Continuing the pattern suggests that the following are true.

$$2^{-1} = \frac{1}{2}$$
$$2^{-2} = \frac{1}{4} = \frac{1}{2^2}$$
$$2^{-3} = \frac{1}{8} = \frac{1}{2^3}$$
$$2^{-4} = \frac{1}{16} = \frac{1}{2^4}$$

A general description of the pattern is simple: $2^{-n} = \frac{1}{2^n}$. That is, 2^{-n} is the reciprocal of 2^n. We call the general property the *Negative Exponent Property*.

Negative Exponent Property

For any nonzero b and all n, $b^{-n} = \frac{1}{b^n}$, the reciprocal of b^n.

Mental Math

Give the area of

a. a square with side $\frac{s}{2}$.

b. a circle with radius $3r$.

c. a rectangle with $\frac{3}{4}x$ and $\frac{8}{3}y$ dimensions.

Notice that even though the exponent in 2^{-4} on the previous page is negative, the number 2^{-4} is still positive. All negative integer powers of positive numbers are positive.

STOP QY

QY

Write 5^{-4} as a simple fraction without a negative exponent.

Example 1

Rewrite $a^7 \cdot b^{-4}$ without negative exponents.

Solution

$$a^7 \cdot b^{-4} = a^7 \cdot \frac{1}{b^4} \qquad \text{Substitute } \frac{1}{b^4} \text{ for } b^{-4}.$$

$$= \frac{a^7}{b^4}$$

Because the Product of Powers Property applies to all exponents, it applies to negative exponents. Suppose you multiply b^n by b^{-n}.

$b^n \cdot b^{-n} = b^{n + -n}$ Product of Powers Property

$= b^0$ Property of Opposites

$= 1$ Zero Exponent Property

To multiply b^n by b^{-n}, you can also use the Negative Exponent Property.

$b^n \cdot b^{-n} = b^n \cdot \frac{1}{b^n}$ Negative Exponent Property

$= 1$ Definition of reciprocal

In this way, the Product of Powers Property verifies that b^{-n} must be the reciprocal of b^n. In particular, $b^{-1} = \frac{1}{b}$. That is, the –1 power (read "negative one" or "negative first" power) of a number is its reciprocal.

Suppose the base b is a fraction, $b = \frac{x}{y}$. Then the reciprocal of b is $\frac{y}{x}$. Consequently, this gives us a different form of the Negative Exponent Property that is more convenient when the base is a fraction. The simplest way to find the reciprocal of a fraction $\frac{a}{b}$ is to invert it, producing $\frac{b}{a}$.

Negative Exponent Property for Fractions

For any nonzero x and y and all n, $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$.

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Example 2

Write each expression without negative exponents.

a. $\left(\frac{5}{4}\right)^{-2}$

b. $\left(\frac{1}{m^2}\right)^{-3}$

Solution

a. Use the Negative Exponent Property for Fractions.

$$\left(\frac{5}{4}\right)^{-2} = \left(\underline{\ ?\ }\right)$$
$$= \underline{\ ?\ }$$

b. Take the reciprocal to the opposite power.

$$\left(\frac{1}{m^2}\right)^{-3} = \left(\frac{?}{1}\right)^{3}$$
$$= (\underline{\ ?\ })^{3}$$
$$= \underline{\ ?\ }$$

Recall the compound interest formula $A = P(1 + r)^t$. In this formula, negative exponents stand for unit periods going back in time.

Example 3

Ten years ago, Den put money into a college savings account at an annual yield of 6%. If the money is now worth \$9,491.49, what was the amount initially invested?

Solution

Here $P = 9{,}491.49$, $r = 0.06$, and $t = -10$ (for 10 years ago).

So, $A = 9{,}491.49(1.06)^{-10} \approx 5{,}300$.

So, Den originally started with approximately \$5,300.

Check Use the Compound Interest Formula. If Den invested \$5,300, he would have $5{,}300(1.06)^{10}$, which equals \$9,491.49. It checks.

Quotient of Powers and Negative Exponents

The last lesson involved fractions in which two powers of the same base are divided. When the denominator contains the greater power, negative exponents can be used to simplify the expression. For example, $\frac{x^5}{x^9} = x^{5-9} = x^{-4}$.

This can be verified using repeated multiplication.

$$\frac{x^5}{x^9} = \frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^4}$$

In this way, you can see again that $b^{-n} = \frac{1}{b^n}$.

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Example 4

Simplify $\frac{5a^4b^7c^2}{15a^{11}b^5c^3}$. Write the answer without negative exponents.

Solution

$\frac{5a^4b^7c^2}{15a^{11}b^5c^3} = \frac{5}{15} \cdot \frac{a^4}{a^{11}} \cdot \frac{b^7}{b^5} \cdot \frac{c^2}{c^3}$ Group factors with the same base together.

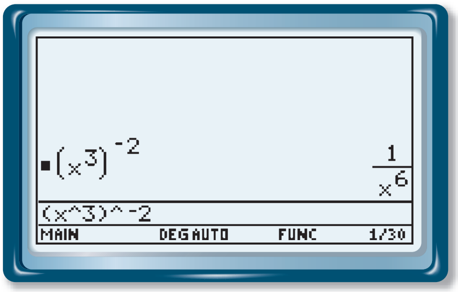
$= \frac{1}{3} \cdot a^{\underline{?}} \cdot b^{\underline{?}} \cdot c^{\underline{?}}$ Quotient of Powers Property

$= \frac{1}{3} \cdot \frac{1}{a^{\underline{?}}} \cdot \frac{b^{\underline{?}}}{1} \cdot \frac{1}{c^{\underline{?}}}$ Negative Exponent Property

$= \underline{?}$ Multiply the fractions.

Applying the Power of a Power Property with Negative Exponents

Consider $(x^3)^{-2}$, a power of a power. Wanda wondered if the Power of a Power Property would apply with negative exponents. She entered the expression into a CAS and the screen below appeared.



This is the answer that would result from applying the Power of a Power Property.

$$(x^3)^{-2} = x^{3 \cdot -2} = x^{-6}$$

Then you can rewrite the power using the Negative Exponent Property.

$$x^{-6} = \frac{1}{x^6}$$

All the properties of powers you have learned can be used with negative exponents. They can translate an expression with a negative exponent into one with only positive exponents.

Example 5

Simplify $(y^{-4})^2$. Write without negative exponents.

Solution

$(y^{-4})^2 = y^{-8}$ Power of a Power Property

$= \frac{1}{y^8}$ Negative Exponent Property

Questions

COVERING THE IDEAS

1. **Fill in the Blanks** Complete the last four equations in the pattern below. Then write the next equation in the pattern.

$$3^4 = 81$$
$$3^3 = 27$$
$$3^2 = 9$$
$$3^1 = 3$$
$$3^0 = 1$$
$$3^{-1} = \underline{\ ?\ }$$
$$3^{-2} = \underline{\ ?\ }$$
$$3^{-3} = \underline{\ ?\ }$$
$$3^{-4} = \underline{\ ?\ }$$

In 2–5, write as a simple fraction.

2. 7^{-2}
3. 5^{-3}
4. $\left(\frac{2}{3}\right)^{-1}$
5. $(y^6)^{-4}$

In 6–9, write as a negative power of an integer.

6. $\frac{1}{36}$
7. $\frac{1}{81}$
8. 0.1
9. 0.0001

10. Eight years ago, Abuna put money into a college savings account at an annual yield of 5%. If there is now $7,250 in the account, what amount was initially invested? Round your answer to the nearest penny.

11. Rewrite each expression without negative exponents.
    a. w^{-1}
    b. $w^{-1}x^{-2}$
    c. $w^{-1}y^3$
    d. $5w^{-1}x^{-2}y^3$

In 12–14, write each expression without negative exponents.

12. $9^2 \cdot 9^{-2}$
13. $n^a \cdot n^{-a}$
14. $(m^{-5})^3$

15. Simplify $\frac{32a^8bc^3}{8a^6b^4c}$. Write without negative exponents.

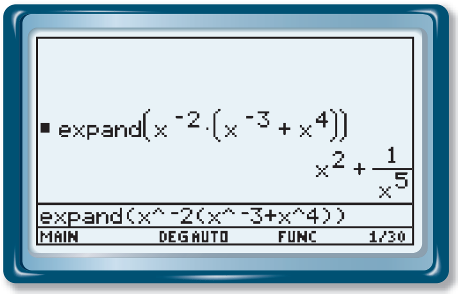
16. a. Graph $y = 2^x$ when the domain is {–4, –3, –2, –1, 0, 1, 2, 3, 4}.
    b. Describe what happens to the graph as x decreases.

17. Graph $y = 10^x$ as on page 457. Describe what happens as x goes from 0 to –12.

APPLYING THE MATHEMATICS

18. If the reciprocal of $a^{-12}b^5$ is $a^n b^m$, find m and n.

19. Use properties of algebra to justify the answer shown on the CAS screen below.

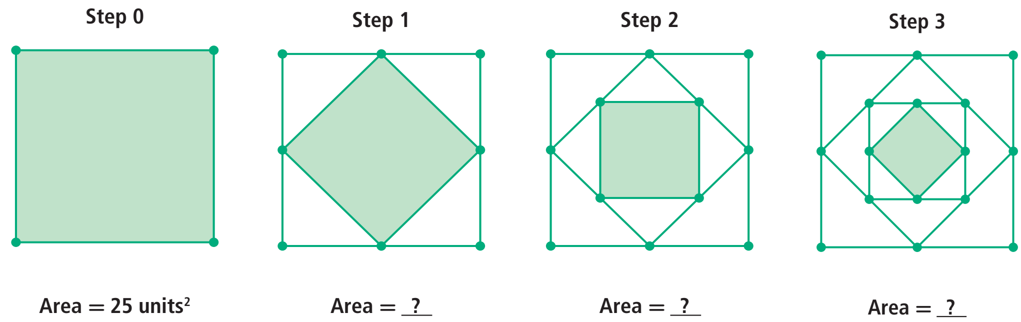


In 20 and 21, solve and check each equation.

20. $7^a \cdot 7^3 = 7^{-6}$

21. $5^m \cdot \frac{1}{25} = 5^{-3}$

22. Suppose you draw a square with area 25 square units and connect the midpoints of each side to create a smaller square inside the original. A sequence of successively smaller squares may be created by repeating the process with the most recently created square. The shaded regions show squares in the sequence.



Write the area of the shaded square for each step as 25 times a power of 2.

a. Step 1
b. Step 2
c. Step 10
d. Step n

REVIEW

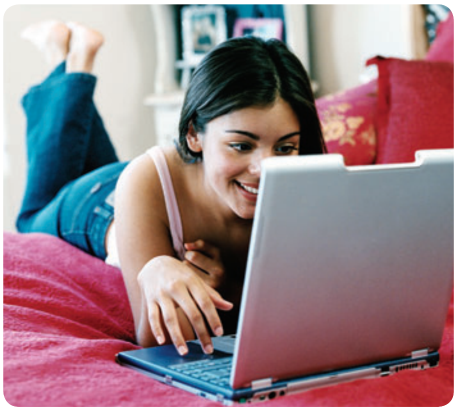
In 23–25 first simplify. Then evaluate when $a = 2$ and $b = 5$. (Lessons 8-3, 8-2)

23. $\frac{a^2 \cdot a^5 \cdot a^3}{a^4}$

24. $(b^2a^{-2})^3$

25. $(2b^3)^a$

26. Some people use randomly generated passwords to protect their computer accounts. Suppose a Web site uses random passwords that are six characters long. They allow only lower-case letters and the digits 0 through 9 to be used. **(Lessons 8-1, 5-6)**
    - **a.** What is the total number of possible passwords?
    - **b.** Jacinta forgot her password. What is the probability that she will guess her password correctly on the first try?
    - **c.** Myron says there would be more possibilities available if the site switched to passwords four characters long but allowed the use of upper-case letters as well. Is Myron correct? Why or why not?



Nearly 49 million laptop computers were sold worldwide in 2004, almost double the number sold in 2000.

Source: *USA Today*

27. Tyra is learning addition and multiplication. For practice, Tyra's teacher gives her a whole number less than 13. Tyra then multiplies the number by 8, adds 25, and states her answer. **(Lessons 7-6, 7-5)**
    - **a.** Describe the situation with function notation, letting x be the number Tyra is given and $m(x)$ the number Tyra states.
    - **b.** What is the domain of the function you wrote?
    - **c.** What are the greatest and least values the function can have?

EXPLORATION

28. Objects in the universe can be quite small. Do research to find objects of the following sizes.
    - **a.** 10^{-3} meter
    - **b.** 10^{-6} meter
    - **c.** 10^{-9} meter
    - **d.** 10^{-12} meter

QY ANSWER

$\frac{1}{625}$