Lesson

5-9

Proportions

▶ **BIG IDEA** Proportions can be solved algebraically using the Means-Extremes Property.

In his 1859 autobiography, Abraham Lincoln wrote about his childhood in Indiana. "There were some schools, so called; but no qualification was ever required of a teacher beyond 'readin, writin, and cipherin' to the Rule of Three. . . . Of course when I came of age I did not know much. Still somehow, I could read, write, and cipher to the Rule of Three; but that was all." After his short stint at school, Lincoln went on to teach himself algebra and geometry from books.

The "Rule of Three" refers to a method of solving a *proportion*, a sentence such as "4 is to 6 as 3 is to __?__." It is equivalent to solving $\frac{4}{6} = \frac{3}{x}$. In a simple proportion such as this, whenever you know three out of the four numbers, you can determine the fourth. The Rule of Three is a method that dates from ancient times and is not usually taught today. Today we use algebra, and so are able to solve this and more complicated proportions.

Solving Proportions

A **proportion** is a statement that two ratios are equal. Thus any equation of the form $\frac{a}{b} = \frac{c}{d}$ is a proportion. This equation is sometimes written $a : b = c : d$. Because a and d are at the two ends of this statement, a and d are called the **extremes**. Because b and c are in the middle, b and c are called the **means.**

Example 1

Complete the sentence; "4 is to 6 as 3 is to __?__."

Solution Let x be the unknown number.

$\frac{4}{6} = \frac{3}{x}$ Write the sentence as a proportion.

$\frac{4}{6} \cdot 6x = \frac{3}{x} \cdot 6x$ Multiply each side of the equation by $6x$.

$\frac{4}{\cancel{6}} \cdot \cancel{6}x = \frac{3}{\cancel{x}} \cdot 6\cancel{x}$ Simplify each side.

$4x = 18$ Simplify.

$x = \frac{18}{4} = 4.5$ Divide each side by 4 and simplify.

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The Rail Splitter, a painting by J. L. G. Ferris of a young Abraham Lincoln splitting logs

Check Substitute 4.5 for x in the proportion $\frac{4}{6} = \frac{3}{x}$. Does $\frac{4}{6} = \frac{3}{4.5}$? Yes, in lowest terms the left side equals $\frac{2}{3}$, and the right side equals $\frac{30}{45}$ or, in lowest terms, $\frac{2}{3}$.

The Means-Extremes Property

Look at Step 4 in the solution to Example 1. The left side, $4x$, is the product of the extremes, 4 and x of the original proportion. The right side, 18, is the product of the means 6 and 3. The general pattern is that the product of the means is equal to the product of the extremes. Algebra explains why this is true. Consider any proportion $\frac{a}{b} = \frac{c}{d}$.

$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$ Multiply both sides of the equation by bd to clear the fractions.

$\frac{a}{\cancel{b}} \cdot \cancel{b}d = \frac{c}{\cancel{d}} \cdot b\cancel{d}$ Simplify both sides of the equation.

$ad = bc$ Simplify.

> ### Means-Extremes Property
> For all real numbers a, b, c, and d (with b and d not zero), if $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

Proportions and Statistics

In statistics, the set of individuals or objects you want to study is called the **population** for that study. If you cannot collect data from the entire population, the part studied is called a **sample.** When samples are taken **randomly,** every member of the population has an equal chance of being chosen. Therefore, data from the sample can be used to estimate information about the population.

Activity

Step 1 Fill a large bowl with pennies, beans, popcorn kernels, or something similar. Do not count how many there are.

Step 2 Take out at least 30 pieces. Mark or tag them and then return the pieces to the bowl.

Step 3 Mix the contents of the bowl. Draw out a handful of the pieces. Be sure to get at least one tagged piece.

How many pieces are there in your handful? __?__

How many pieces are tagged? __?__

Step 4 Set up a proportion to estimate how many pieces are in the bowl and solve it.

Step 5 Return all pieces to the bowl and repeat Steps 3 and 4 at least two more times.

Step 6 Compare and contrast your totals. Based on your random samples, do you believe you have a good estimate of how many pieces are in the bowl? Explain why or why not.

In this activity, the method you used to find the total pieces in the bowl is called the **capture-recapture method.** This method has been used to estimate the number of fish in a lake or deer in a forest. In Step 2, you captured pieces and tagged them. Step 3 is where you recaptured the pieces. If the sample in the recapture is chosen randomly, the ratio of tagged pieces to the number of pieces is nearly the same as the percentage of tagged pieces in the entire bowl.

Surveys give information about a part of a population. Proportions are used to extend that information to a larger group. When we do this, we assume that the ratios are the same for the survey group and the larger population.

A biologist uses a computer while colleagues capture spring-summer Chinook salmon smolts for counting and tagging on the Idaho Salmon River.

Example 2

A survey of 454 undergraduates at the University of Texas in 2003 found that 409 of them used their own computers at home. The actual number of undergraduates at the school was 37,409. Based on the survey, how many undergraduates were expected to use computers at home?

Solution 1 Let x be the number of 2003 undergraduates using a personal computer. Set up a proportion with equal ratios that compares undergraduates with computers to the total number of undergraduates.

$$\frac{\text{number of undergraduates with computers in survey}}{\text{total number of undergraduates surveyed}} = \frac{\text{number of total undergraduates with computers}}{\text{total number of undergraduates}}$$

$$\frac{409}{454} = \frac{x}{37,409}$$

$454x = 409 \cdot 37,409$ Means-Extremes Property

$454x = 15,300,281$ Simplify.

$x \approx 33,701$ Multiply each side by $\frac{1}{454}$.

About 33,701 undergraduates would be expected to have computers at home in 2003.

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Solution 2 Let x be the number of 2003 undergraduates using their own computer. Use ratios that compare undergraduates with computers to the total number of undergraduates.

$$\frac{\text{number of undergraduates with computers in survey}}{\text{total number of undergraduates with computers}} = \frac{\text{total undergraduates surveyed}}{\text{total number of undergraduates}}$$

$$\frac{409}{x} = \frac{454}{37{,}409}$$

$409 \cdot 37{,}409 = 454x$	Means-Extremes Property
$15{,}300{,}281 = 454x$	Simplify.
$33{,}701 \approx x$	Multiply each side by $\frac{1}{454}$.

About 33,701 undergraduates were expected to have computers at home in 2003.

The equations that were written for the two solutions are different. But notice that when the Means-Extremes Property is used, the resulting equations are the same.

🛑 **QY**

Some proportions contain algebraic expressions.

▶ **QY**

Emilio knows he can do 11 pushups in 15 seconds.

a. If his wrestling coach times him for a minute, at this rate how many pushups could he do?

b. At this rate, how long will it take Emilio to do 74 pushups?

c. Why is the answer to Part b likely an underestimate?

Example 3

Two candidates, A and B, ran in a village election. Candidate A received 450 more votes than Candidate B. Their vote counts were in the ratio of 8 to 3. How many votes did each candidate get?

Solution Let $x =$ number of votes Candidate B received. Because Candidate A received 450 more votes than Candidate B, the expression $x + 450$ is the number of votes Candidate A received.

$$\frac{\text{number of votes for A}}{\text{number of votes for B}} = \frac{x + 450}{x} = \frac{8}{3}$$

$3(x + 450) = x \cdot 8$	Means-Extremes Property
$3x + 1{,}350 = 8x$	Distribute.
$1{,}350 = 5x$	Add $-3x$ to both sides.
$270 = x$	Divide both sides by 5.

Because $x = 270$, Candidate B got 270 votes. Candidate A got $x + 450$, or $270 + 450 = 720$ votes.

Check Is the ratio of A's votes to B's votes equal to $\frac{8}{3}$?

Is $\frac{720}{270} = \frac{8}{3}$?

$\frac{8}{3} = \frac{8}{3}$ Yes, it checks.

Questions

COVERING THE IDEAS

In 1 and 2, complete the sentence.

1. 4 is to 12 as 18 is to __?__.

2. 5 is to 13 as 17 is to __?__.

3. What is a proportion?

In 4 and 5, a proportion is given.

a. Use the Means-Extremes Property to solve the proportion.

b. Check your work.

4. $\frac{n}{4} = \frac{20}{48}$

5. $\frac{-28}{21} = \frac{64}{p}$

6. On a map of Spain, 3 centimeters represents 200 kilometers.

    a. Seville and Madrid are approximately 417 kilometers apart. How far apart would they be on the map?

    b. If Barcelona and Madrid are 9.4 centimeters apart on a map, about how many kilometers apart are they?

7. After soccer's World Cup was held in Germany in 2006, the national tourist board surveyed about 1,300 foreign visitors who attended. Of them, approximately 1,200 responded that the World Cup had been a great event.

    a. Based on the survey, about how many of the 2 million foreign visitors felt the World Cup was a great event?

    b. How many foreign visitors felt the World Cup was *not* a great event?

8. In a capture-recapture study, suppose 60 deer in a forest are tagged. On the recapture, 52 deer are caught, of which 10 are found to have been tagged. Estimate the number of deer in the forest.

9. In 1995, scientists began restoring gray wolves to Yellowstone National Park. The recovery plan called for introducing 10 breeding pairs of gray wolves each year for three years. Suppose that in 2000, the scientists recaptured 14 wolves and 3 had tags. What was the estimated population of gray wolves in Yellowstone in 2000?

10. Two numbers are in the ratio of 9 : 5. One number is 76 greater than the other. What are the numbers?

In 11 and 12, solve the proportion.

11. $\frac{4m - 1}{7} = \frac{m + 2}{2}$

12. $\frac{5}{12} = \frac{2p - 3}{3p + 5}$

France and South Korea square off in group play during the 2006 World Cup in Berlin, Germany.

APPLYING THE MATHEMATICS

In 13–15, solve the proportion for the indicated variable. No variables equal 0.

13. $\frac{2}{3} = \frac{b}{c}$ for c **14.** $\frac{4x}{w} = \frac{3}{m}$ for w **15.** $\frac{a}{b} = \frac{x}{y}$ for a

16. A baseball team plays 2 innings in 25 minutes. At this rate, how many minutes will a 9-inning game take?

17. During the first 7 days of November, Gabby used her cell phone for 133 minutes. At this rate, how many minutes will she talk during the entire month?

18. Kauai, Hawaii, is considered the rainiest place on Earth. In an average week, 9.1 inches of rain falls on the island. If you are on Kauai for 2 days, how much rain would you expect to fall during your stay?

19. The target heart rates for 22-year-old females exercising in the "fitness zone" is 122–143 beats per minute. Annie, a 22-year-old female, regularly checks her pulse rate while exercising. She found that her heart beats 19 times in 10 seconds.

   a. At this rate, how many times does Annie's heart beat in 60 seconds?

   b. Is Annie in her target heart rate zone?

20. The Havalot family bought a 26-inch and a 50-inch plasma TV. The total cost of the two televisions was $4,600. If the ratio of the prices was 6 : 17, how much did each TV cost?

21. A useful baseball statistic is a pitcher's earned run average (ERA), which is a measure of the average number of runs a pitcher allows during 9 innings. Suppose a pitcher has an ERA of 3.33 and has pitched 150 innings. How many earned runs has he allowed during those innings?
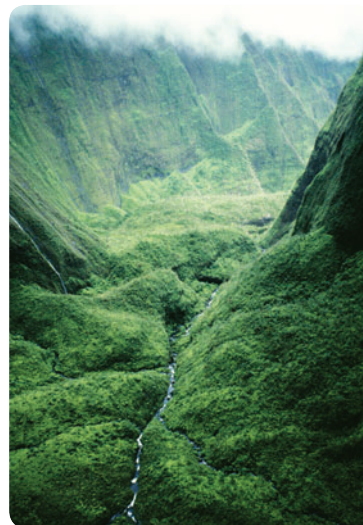
Mt. Waialeale on the island of Kauai in Hawaii (3,000 feet high) is the wettest spot on Earth, averaging about 460 inches of rain per year.

Source: www.infoplease.com

REVIEW

22. A raft that is a rectangle 8 feet by 12 feet is in a circular pool that is 40 feet in diameter. If a watch is at the bottom of the pool, what is the probability it is under the raft? (**Lesson 5-8**)

23. Consider the following situation. A bowl contains 8 green beads, 4 red beads, 11 blue beads, and 6 black beads. One bead from the bowl is then chosen at random. (**Lesson 5-6**)

   a. Find the probability of choosing a red bead.

   b. Find the probability of choosing a black or blue bead.

   c. Find the probability of choosing a bead that is *not* green.

24. Square I has sides of length a and Square II has sides of length $3a$, as shown at the right. (**Lesson 5-5**)

    a. Find the ratio of a side of Square II to a side of Square I.

    b. Find the areas of Square I and Square II.

    c. Find the ratio of the area of Square II to the area of Square I.

Square I   **Square II**
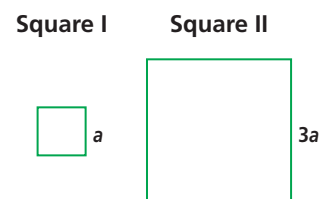
25. A formula to find the sum S of the measures of the interior angles of a polygon is $S = 180(n - 2)$, where n is the number of sides of the polygon. (**Lesson 4-7**)

    a. Solve this formula for n.

    b. If the sum of the measures of the interior angles of a polygon is $1{,}260°$, find the number of sides of the polygon.

26. **Skill Sequence** Find each reciprocal. (**Lesson 2-8**)

    a. $\dfrac{5}{9}$ **b.** $\dfrac{5x}{9}$ **c.** $\dfrac{-5x}{9}$

27. **a.** The following number puzzle deals with your seven-digit phone number, not including the area code.

Step 1 Write down the first 3 digits of your phone number.
Step 2 Multiply this by 80.
Step 3 Add 1.
Step 4 Multiply this by 250.
Step 5 Add the last four digits of your phone number.
Step 6 Add the last four digits of your phone number again.
Step 7 Subtract 250.
Step 8 Divide by 2.

    What is your result?

    b. Use algebra to explain your answer to Part a.
    (**Lesson 2-3**)

EXPLORATION

28. The tallest person ever measured was Robert Wadlow, who was 8 feet 11.1 inches tall. At 13 years of age, he was 7 feet 4 inches tall. Because schools are constructed for a much shorter person, many things were too small for him. Assume that schools are designed for a person who is up to 6 feet tall. Measure the dimensions of five things that you see in school every day. How big would these objects be if they were made proportionally to fit Robert Wadlow when he was 13 years old?

QY ANSWERS

a. 44

b. about 101 seconds

c. As Emilio does more push-ups, he probably gets tired and slows down, making his rate change and the time it takes to do push-ups longer.