Lesson
5-3
Rates

▶ **BIG IDEA**   The quotient of two measures with different units is a rate.

What Is a Rate?

In Lesson 5-2, 32 ounces of orange juice were split evenly among 5 people. By doing the division with the units left in, we see that the answer is a *rate*.

$$\frac{32 \text{ oz}}{5 \text{ people}} = \frac{6.4 \text{ oz}}{\text{person}} = 6.4 \text{ oz/person} = 6.4 \text{ ounces per person}$$

Every rate consists of a number and a *rate unit.* You may see rate units expressed using a slash "/" or a horizontal bar "——". The slash and the bar are read "per" or "for each." The rate unit $\frac{\text{oz}}{\text{person}}$ is read "ounces per person."

In general, a **rate** is the quotient of two quantities with different units.

How Are Rates Calculated?

Since rates are quotients, rates are calculated by dividing.

> ### Example 1
>
> **Tanya and Gary drove 400 miles in 8 hours during a trip. What was their average speed?**
>
> **Solution 1**   Divide the distance in miles by the time in hours.
>
> $$\frac{400 \text{ mi}}{8 \text{ hr}}$$
>
> Separate the measurement units from the numerical parts.
>
> $$\frac{400 \text{ mi}}{8 \text{ hr}} = \frac{400}{8}\frac{\text{mi}}{\text{hr}} = 50 \text{ miles per hour}$$
>
> They were traveling at an average speed of 50 miles per hour.
>
> **Solution 2**   You could also divide the time by the distance.
>
> $$\frac{8 \text{ hours}}{400 \text{ miles}} = \frac{8}{400}\frac{\text{hr}}{\text{mi}} = \frac{1}{50} \text{ hour per mile}$$
>
> This means that on the average, it took them $\frac{1}{50}$ of an hour to travel each mile.

In Example 1, the first solution gives the rate in *miles per hour.* The second solution gives the rate in *hours per mile,* or how long it takes to travel one mile. These are **reciprocal rates.** Notice that $\frac{1}{50}$ of an hour is $\frac{1}{50} \cdot 60$ min, or 1.2 minutes. In other words, it takes a little over a minute to go one mile. Either rate is correct. The one to use depends on the situation in which you use it.

Rates and Negative Numbers

Rates can be positive or negative quantities.

Example 2

a. If the temperature rises from 70 to 85 degrees in 2 hours, what is the change in temperature in degrees per hour?

b. If the temperature goes down from 44 to 32 degrees in 5 hours, what is the rate of temperature change in degrees per hour?

Solutions To find the rate, divide the number of degrees changed by the number of hours.

a. $\text{rate of temperature change} = \dfrac{(85 - 70) \text{ degrees}}{2 \text{ hours}}$

$= \dfrac{15 \text{ degrees}}{2 \text{ hours}}$

$= \text{rise of 7.5 degrees per hour}$

b. $\text{rate of temperature change} = \dfrac{(32 - 44) \text{ degrees}}{5 \text{ hours}}$

$= \dfrac{-12 \text{ degrees}}{5 \text{ hours}}$

$= \text{drop of 2.4 degrees per hour}$

$\text{or } -2.4 \text{ degrees per hour}$

▶ QY1

Find the rate of change of the number of people at a restaurant if it decreased from 161 to 98 people in 45 minutes. (Pay attention to positives and negatives.)

🛑 **QY1**

In Part b of Example 2, the rate –2.4 degrees per hour came from dividing a negative number (–12 degrees) by a positive one (5 hours). The same negative rate can be found by dividing 12 by –5, which would describe the 12-degree rise in temperature that would come from moving backward in time 5 hours.

So $\frac{-12}{5}$ degrees per hour $= \dfrac{-12 \text{ degrees}}{5 \text{ hr}} = \dfrac{12 \text{ degrees}}{-5 \text{ hr}}$.

Here is a way to think of this situation. If you change a numerator or denominator of a fraction to its opposite, then the value of the fraction also changes to its opposite.

Negative Fractions

In general, for all a and b, and $b \neq 0$, $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$.

When *both* the numerator and denominator are changed to their opposites, the value of the fraction is unchanged.

$$\frac{-a}{-b} = \frac{-1 \cdot a}{-1 \cdot b} \quad \text{Multiplication Property of } -1$$
$$= \frac{a}{b} \quad \text{Equal Fractions Property}$$

Fractions with negative numbers are common, as you will see in the next chapter.

Division by Zero and Rates

Consider the rate $\frac{0 \text{ meters}}{10 \text{ seconds}}$, which has 0 in the numerator. This means that you do not travel at all in 10 seconds. So your rate is 0 meters per second. This reinforces that $\frac{0}{10} = 0$. In contrast, try to imagine a rate such as $\frac{10 \text{ meters}}{0 \text{ seconds}}$, which has 0 in the denominator. This would mean you travel 10 meters in 0 seconds. For this to occur, you would have to be in two places at the same time! That is impossible. Rates show that the denominator of a fraction can never be zero.

When a fraction has a variable in its denominator, then the expression is said to be *undefined* for any value of the variable that would make the denominator zero.

GUIDED

Example 3

a. For what value(s) of x is $\frac{x}{x + 4}$ undefined?

b. For what value(s) of x does $\frac{x}{x + 4} = 0$?

Solutions

a. A fraction is undefined whenever its ___?___ is zero.

So $\frac{x}{x + 4}$ is undefined when ___?___ $= 0$.

Solve the equation on the preceding line for x. $x = $ ___?___.

Therefore, $\frac{x}{x + 4}$ is undefined when $x = $ ___?___.

(continued on next page)

b. A fraction equals zero whenever its ____?____ is zero.

So $\frac{x}{x+4} = 0$ when ____?____ $= 0$.

Therefore, $\frac{x}{x+4} = 0$ when $x =$ ____?____.

🛑 QY2

Questions

COVERING THE IDEAS

1. Name all of the rates (including the rate units) in the following paragraph.

   The Indianapolis 500 is one of the most famous auto races in the United States, with hundreds of thousands of people attending annually. Attendees in 2006 paid from \$40 to \$150 per ticket plus \$20 to \$50 per hour to park. Drivers reached speeds of more than 230 mph (354 km/hr) as they raced the 2.5-mile oval track.

2. Give an example of a rate with a rate unit that is not mentioned in the reading of this lesson.

An IRL IndyCar Series car accelerates from 0 to 100 mph in less than three seconds.

Source: www.indy500.com

In 3–5, calculate a rate suggested by the given information.

3. Danielle walked her dog 6 blocks in t minutes.

4. In the last seven days, Salali slept 6.5 hours one night, 7 hours two nights, 7.5 hours two nights, 8 hours one night, and 9.5 hours one night.

5. In 2004, 2.3 billion books were sold in the United States, which had a population of about 296 million people.

6. In playing a video game 4 times, Bailey scored a points, b points, c points, and d points.

   a. Give an expression for her average score.

   b. Bailey's average is a rate. What is the rate unit?

7. Translate the change in time and temperature into positive and negative quantities. Then calculate the rate.

   a. 8 hours ago it was 5 degrees warmer than it is now.

   b. If this rate continues, then 8 hours from now it will be 5 degrees colder.

8. **Multiple Choice** Which of these numbers is *not* equal to the others?

   A $\frac{-153x}{82}$ **B** $-\frac{153x}{82}$ **C** $\frac{-153x}{-82}$ **D** $\frac{153x}{-82}$ **E** $\frac{-(-153)x}{-82}$

In 9–11, an expression is given.

a. **For what values of the variable is the expression equal to zero?**

b. **For what value of the variable is the expression undefined?**

9. $\dfrac{w - 12}{w + 5}$ 10. $\dfrac{17}{m - 4}$ 11. $\dfrac{2 + y}{15}$

APPLYING THE MATHEMATICS

12. When you buy something in quantity, the cost of one item is the *unit cost.* Find the unit cost for each of the following.

a. frozen juice at 3 cans for $5

b. 500 sheets of notebook paper for $2.49

c. x paper clips for $0.69

13. Let $y = \dfrac{1 + x}{2 - x}$. Complete the table at the right.

x	y
−3	?
−2	?
−1	?
0	?
1	?
2	?
3	?

14. A very fast runner can run a half-mile in 2 minutes. Express the average rate in each of these units.

a. miles per minute   b. minutes per mile   c. miles per hour

15. For each state below, find the number of people per square mile to the nearest tenth. This is the state's *population density* for 2005.

a. New Jersey: population = 8.7 million; area = 8,700 square miles

b. Montana: population = 0.9 million; area = 147,000 square miles

16. **Multiple Choice** In t minutes, a copy machine made n copies. At this rate, how many copies per second does the machine make?

A $\dfrac{n}{60t}$ B $\dfrac{60t}{n}$ C $\dfrac{60n}{t}$ D $\dfrac{t}{60n}$

17. The Talkalot cell phone company sells a pay-as-you-go phone with 700 minutes for $70.

a. What is the rate per minute?

b. What is the rate per hour?

c. Elizabeth buys a phone and talks for m minutes. What is the value of the phone now?

REVIEW

In 18–20, simplify the expression. (Lessons 5-2, 5-1, 2-2)

18. $\dfrac{4xy}{-3y^2} \cdot \dfrac{-6y}{5x^2}$

19. $\dfrac{ab}{21} \div \dfrac{a}{4b}$

20. a. $\dfrac{-8n}{3} \cdot \dfrac{8n}{3}$ b. $\dfrac{-8n}{3} + \dfrac{8n}{3}$

21. Alice has n pounds of bologna. If she uses $\frac{1}{8}$ pound of bologna to make one bologna sandwich, how many bologna sandwiches can Alice make? (Lesson 5-2)

In 22–25, simplify the expression. (Lesson 5-2)

22. $\dfrac{\frac{9}{4}}{5}$

23. $\dfrac{-6}{7} \div \dfrac{-6}{5}$

24. $\dfrac{a}{4b} \div \dfrac{b}{6}$

25. $\dfrac{2d}{3} \div 5d$

26. **a.** Draw a square with side of length x.

  b. Shade or color the diagram to show $\dfrac{x}{2} \cdot \dfrac{3x}{4}$.

  c. What is the result of the multiplication? (Lesson 5-1)

27. *Skill Sequence* Solve each inequality. (Lessons 4-5, 3-7)

  a. $-y > 10$

  b. $-5x < 10$

  c. $-2A + 3 \le 10$

  d. $-9B + 7 \ge 10 + 3B$

28. **a.** How many seconds are in one day?

  b. A second is what fraction of a day? (Previous Course)

EXPLORATION

29. Use the Internet, an almanac, or some other source to find the estimated current U.S. national debt and an estimate of the U.S. population. Then calculate the average debt per capita. (The phrase *per capita* means "per person.")

QY ANSWERS

1. a decrease of 1.4 people/min

2. **a.** $k = 55$

  b. $k = 3$