Lesson 4-9

Solving Absolute Value Equations and Inequalities

BIG IDEA Inequalities with $|ax + b|$ on one side and a positive number c on the other side can be solved by using the fact that $ax + b$ must equal either c or $-c$.

A school carnival had a "Guess the Number" booth featuring a jar full of pennies. Whoever guessed closest to the actual number of pennies would win a prize. Only the principal knew there were 672 pennies in the jar. When the prize was announced, the winner was off by 9 pennies. How many pennies did the winner guess?

The winning guess deviated from the actual number of pennies by 9. This does not say if the guess was too high or too low. It could have been either $672 + 9 = 681$ or $672 - 9 = 663$. These two possibilities are shown on the number line at the right.

The two numbers 663 and 681 are the solutions of $|n - 672| = 9$. The expression $|n - 672|$ is the absolute deviation of the guess n, from the actual number of pennies, 672. In this case, $|n - 672| = 9$ and the solutions to the equation are 663 and 681.

The equation $|n - 672| = 9$ is of the form $|ax + b| = c$, with $a = 1$, n in place of x, $b = -672$, and $c = 9$. All equations of this form can be solved using what you know about linear equations and compound sentences.

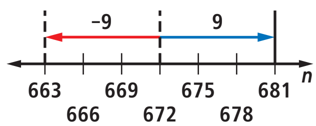
Mental Math

When $a > 0$, determine if the following are positive or negative.

a. $\left(-\frac{a}{2}\right)^2$

b. $(-5a)^3$

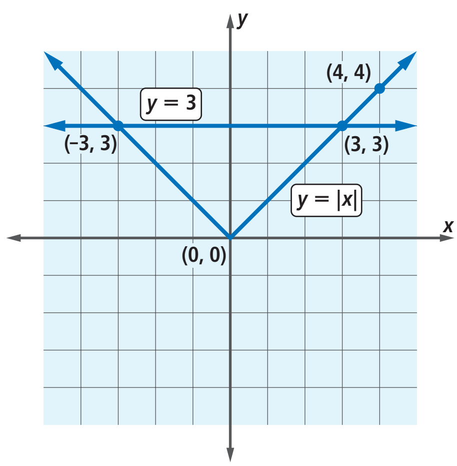
c. $-(-0.9a)^2$



Solving $|x| = a$

Remember that $|x|$ is the distance of x from 0 on a number line. At the right is a table of some pairs of values of x and $|x|$ and the graph of $y = |x|$. Also, the line $y = 3$ is graphed.

x	$\lvert x\rvert$
−4	4
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3
4	4



The following conclusions about the equation $|x| = c$ can be made from the table and graph.

- $|x|$ is never negative, so there are no solutions to $|x| = -10$ or to any other equation of the form $|x| = c$ when c is negative.
- $|x| = 0$ only when $x = 0$.
- $|x| = 3$ has two solutions, 3 and –3. The two solutions can be seen in the table in the rows where $|x| = 3$. Also, the graph of the horizontal line $y = 3$ intersects $y = |x|$ in two points, where $x = 3$ and $x = -3$. Therefore, there are always two solutions to $|x| = c$ when c is positive.

Solutions to $|x| = c$

- When c is positive, then there are two solutions to $|x| = c$, namely c and $-c$.
- When c is negative, then there are no solutions to $|x| = c$.
- When c is zero, then $|x| = 0$ and there is one solution: $x = 0$.

STOP **QY1**

QY1

Find all the solutions to each equation.

a. $|t| = 15$

b. $|u| = -88.2$

c. $|v| = 0$

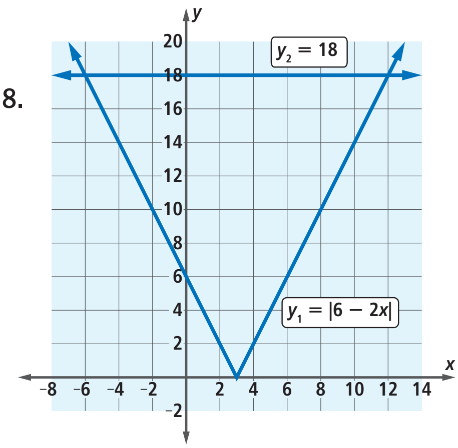
Solving $|ax + b| = c$

The above ideas apply to any equation where there is an absolute value of an expression on one side and a number on the other.

GUIDED

Example 1

Consider the graph, which shows $y_1 = |6 - 2x|$ and $y_2 = 18$. Use the graph and use algebraic properties to solve $|6 - 2x| = 18$.



Solution 1 Use the graph.

The points of intersection of the two graphs are __?__ and __?__.
The x-coordinates of these points of intersection are __?__ and __?__.

The solutions to $|6 - 2x| = 18$ are __?__ and __?__.

Solution 2 Use algebraic properties.

Ask yourself: What numbers have absolute value 18?
$6 - 2x =$ __?__ or $6 - 2x =$ __?__

Solve this compound sentence as you did in Lesson 4-8.

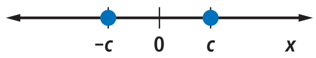
6	$= 2x +$ __?__	or	6	$= 2x +$ __?__
__?__	$= 2x$	or	__?__	$= 2x$
__?__	$= x$	or	__?__	$= x$

You should have the same answers from both Solutions 1 and 2.

Solving $|x| < c$ and $|x| > c$

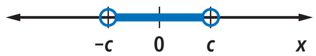
When c is positive, the two solutions to $|x| = c$ can be represented on a number line. They are the points at a distance c from 0.

$|x| = c$ if and only if $x = c$ or $x = -c$.



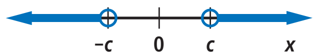
The points closer to 0 than c are the solutions to the inequality $|x| < c$. They can be described by the interval $-c < x < c$. For example, the solutions to $|x| < 3$ can be described by the double inequality $-3 < x < 3$.

$|x| < c$ if and only if $-c < x < c$.



The solutions to the inequality $|x| > 3$ are the points whose distance from 0 is greater than 3. The graph of these points has two parts and is described by the compound inequality $x < -3$ or $x > 3$.

$|x| > c$ if and only if $x < -c$ or $x > c$.



Solutions to $|x| < c$ when c is positive

$|x| < c$ if and only if $-c < x < c$.

Solutions to $|x| > c$ when c is positive

$|x| > c$ if and only if $x < -c$ or $x > c$.

STOP **QY2**

▶ QY2

Describe all solutions to $|d| > 0.9$ using a compound inequality without the absolute value symbol.

Solving $|ax + b| < c$, $|ax + b| > c$

To solve these inequalities, think of the simpler inequalities $|x| > c$ and $|x| < c$.

GUIDED

Example 2

Solve $|8x + 24| \le 40$.

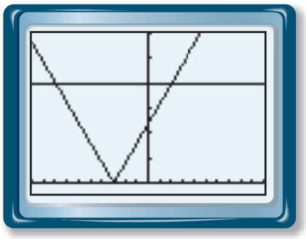
Solution 1 Think: $|x| < a$ means $-a < x < a$.

$\lvert 8x + 24\rvert \le 40$	Write the inequality.
$-40 \le 8x + 24 \le 40$	$\lvert x\rvert < a$ means $-a < x < a$.
$\underline{\ ?\ } \le 8x \le \underline{\ ?\ }$	Add −24 to each side.
$\underline{\ ?\ } \le x \le \underline{\ ?\ }$	Divide both sides by 8.

Solution 2 Use a graph or table.

Ask yourself: When is $|8x + 24|$ below or at 40 on the graph? When is $|8x + 24|$ less than or equal to 40 in the table?

x	$\lvert 8x + 24 \rvert$
−10	?
−8	?
−6	?
−4	?
−2	?
0	?
2	?
4	?
6	?



$-10 \le x \le 10$; x scl = 1
$0 \le y \le 60$; y scl = 10

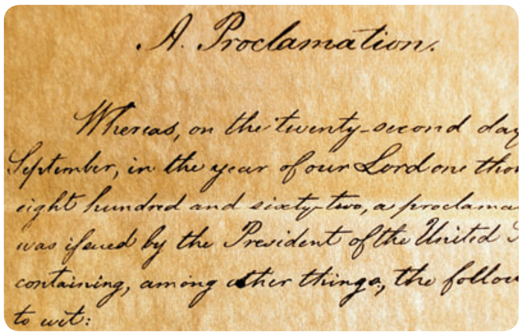
So the solution to $|8x + 24| \le 40$ is _?_ $\le x \le$ _?_.

Questions

COVERING THE IDEAS

1. **Multiple Choice** You are asked for the year of the Emancipation Proclamation in the United States on a test. The correct answer is 1863. You guessed g and you were off by 4 years. What equation's solution gives the possible values of g?

   A $|1863 - 4| = g$ B $|g| = 1863 - 4$
   C $|g - 1863| = 4$ D $|g - 4| = 1863$



In September 1862, Abraham Lincoln called on the seceded states to return to the Union or have their slaves declared free. When no state returned, he issued the proclamation on January 1, 1863.

Source: Britannica

2. Determine whether the number is a solution to the equation $60 = |n - 90|$.

   a. 30 b. −30 c. 150 d. −150

In 3–6, find all solutions in your head.

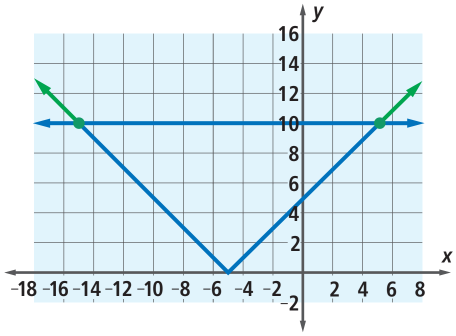
3. $|A| = 6$ 4. $|B| = -600$ 5. $|C| = 0$ 6. $5|D| = 40$

7. Use the table at the right to solve each sentence.

   a. $|2x - 3| = 7$
   b. $|2x - 3| < 7$
   c. $|2x - 3| > 7$

x	$Y_1 = \lvert 2x - 3 \rvert$	$Y_2 = 7$
−5	13	7
−4	11	7
−2	7	7
0	3	7
1	1	7
2	1	7
5	7	7
6	9	7

8. **Fill in the Blanks** The sentence $|ax + b| = 15$ is equivalent to the compound sentence $\underline{\quad ? \quad}$ or $\underline{\quad ? \quad}$.



9. **Multiple Choice** The green portion of the graph can be used to find the solution to which of the following?

A $|x + 5| = 10$ B $|x + 5| \leq 10$

C $|x + 5| \geq 10$ D $|x + 5| > 10$

10. **Fill in the Blanks** $|x - 4| = 2.3$ means the distance between $\underline{\quad ? \quad}$ and $\underline{\quad ? \quad}$ on a number line is $\underline{\quad ? \quad}$.

In 11–16, solve the sentence.

11. $|a + 10| = 12$

12. $|42 - 3b| = 45$

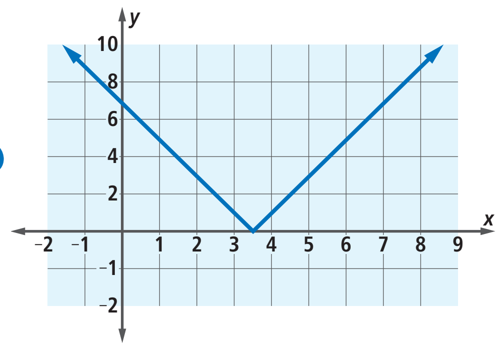
13. $|8 - c| < 9$

14. $|10d + 0.3| > 5.6$

15. $\frac{1}{2} \leq \left|\frac{5}{8}g + \frac{3}{4}\right|$

16. $|h + 11| - 3 \leq 0$

17. Use the graph of $y = |7 - 2x|$ at the right to solve the sentence.



a. $|7 - 2x| = 9$ b. $|7 - 2x| = 0$

c. $|7 - 2x| = -2$ d. $|7 - 2x| > 5$

APPLYING THE MATHEMATICS

18. Let $|5x + 2| = m$. Find a value of m so that the absolute value equation has the given number of solutions.

a. two solutions b. one solution c. no solutions

19. A box of Wheat-Os breakfast cereal says that it contains 24 ounces. However, because the machinery that fills the boxes cannot be exactly precise, they can be from $\frac{1}{8}$ ounce below to 1 ounce above this weight.

a. Graph the possible number of ounces of Wheat-Os.

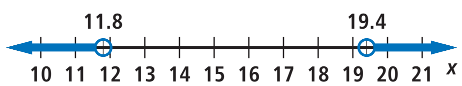
b. Write an absolute value inequality to show this amount.

20. It is recommended that teenagers get 8.5 ± 0.7 hours of sleep.

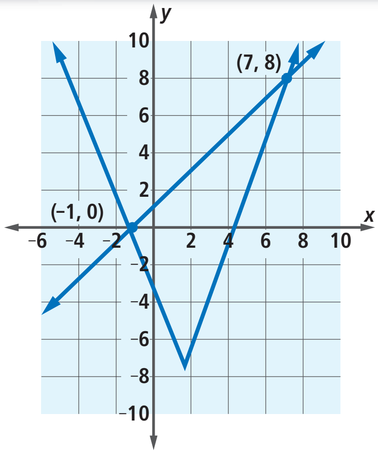
a. Write a double inequality to express the recommended amount of sleep.

b. Write an absolute value inequality for the recommended amount of sleep.

21. Write a single inequality for the graph below.



22. At the right is the graph that Zoey used to solve $|3x - 5| - 8 < x + 1$. Use her graph and intersection points to give the solution.



REVIEW

23. Because of traffic, Nina's drive to school takes anywhere from 12 to 20 minutes. **(Lesson 4-8)**
   a. Write an inequality expressing the time it takes Nina to get to school.
   b. Write an expression of the form $a \pm d$ to represent the interval in Part a.

In 24 and 25, solve the compound inequality and graph the solution. (Lesson 4-8)

24. $4a - 7 \leq 17$ and $14 - a > -5a + 3$

25. $2(m - 2) \geq 7m + 6$ or $4m - 7 > 30$

26. Consider the equations $y = 5x + 9$ and $3x - y = a - 2x$. Find a value of a so that the two lines do not intersect. **(Lesson 4-7)**

27. Write equations for the horizontal and vertical lines that go through the point $\left(-4.25, \frac{9}{11}\right)$. **(Lesson 4-2)**

28. Gail had \$11.50 to spend on snacks for herself and her friends. She wanted to buy as many 85 cent energy bars as she could, in addition to 3 boxes of popcorn at 75 cents each and 3 juice bottles at \$1.50 each. Set up an inequality and solve by clearing decimals to find the number of energy bars she can buy. **(Lesson 3-8)**

EXPLORATION

29. You learned that absolute value equations can have 0, 1, or 2 solutions.
   a. Write an absolute value inequality whose solution is all real numbers.
   b. Write an absolute value inequality that has no solutions.

30. Solve $|x - 2| = |x| - 2$.

QY ANSWERS

1a. $t = 15$ or $t = -15$

b. no solution

c. $v = 0$

2. $d < -0.9$ or $d > 0.9$