Lesson 11-7 Finite Differences

Vocabulary

method of finite differences

BIG IDEA Given n points in the plane, no two of which have the same first coordinate, it is possible to determine whether there is a polynomial function of degree less than $n - 1$ that contains all the points.

Mental Math

Factor.

a. $c^{14} - 100$

b. $r^2 - 10r + 25$

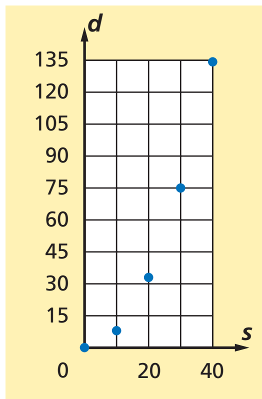
c. $\pi x^4 - 13\pi x^3$

d. $4p^2 + 12pq + 9q^2$

When you find an equation of a line through two data points, you have an *exact model* for the data. When you find an equation for a regression line through a set of data points that are roughly linear, you have an *approximate model* for the data. Similarly, you can find exact and approximate quadratic and exponential models. This lesson is about finding exact polynomial models through data points.

Consider the data points and their scatterplot at the right. The graph looks like part of a parabola, perhaps with its vertex at the origin, so it is reasonable to think that a quadratic polynomial function models these data. But the graph also looks somewhat like an exponential function, translated down one unit.

s	0	10	20	30	40
d	0	8	33	75	134



Differences between Values of Polynomial Functions

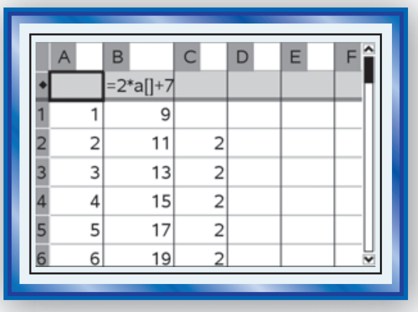
It is possible to determine that a quadratic function is an exact model for these data. The determination relies on finding differences between certain values of the function.

Consider the spreadsheet at the right. Columns A and B show values of x and y, respectively, for the linear function with equation $y = 2x + 7$ when $x = 1, 2, 3, 4, 5$, and 6.

The values in the cells of column C are the differences between consecutive values in the cells of column B.

$$C2 = B2 - B1 = 11 - 9 = 2;$$
$$C3 = B3 - B2 = 13 - 11 = 2;$$
$$C4 = B4 - B3 = 15 - 13 = 2; \text{ and so on.}$$

Notice that all these differences are 2, which is the slope of the line with equation $y = 2x + 7$. So, the spreadsheet shows the constant increase of the linear polynomial $2x + 7$.



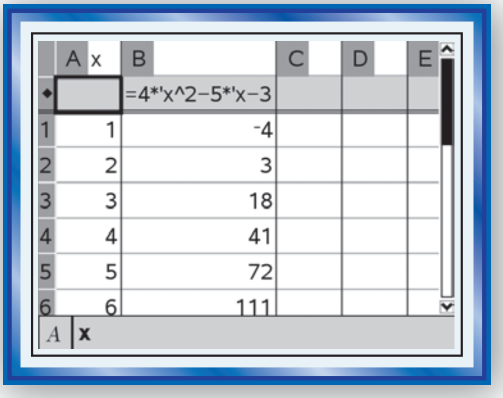
	A	B	C	D	E	F
◆		=2*a[]+7				
1	1	9				
2	2	11	2			
3	3	13	2			
4	4	15	2			
5	5	17	2			
6	6	19	2			

Activity 1

Step 1 **a.** Make a spreadsheet to show x- and y-values for the quadratic polynomial function with equation $y = 4x^2 - 5x - 3$, for $x = 1$ to 7. The first six rows of our spreadsheet are shown at the right.

b. Define a third column showing the difference between consecutive y-values.

c. Define a fourth column showing the difference between consecutive cells of the third column.



	A x	B	C	D	E
◆		=4*'x^2−5*'x−3			
1	1	-4			
2	2	3			
3	3	18			
4	4	41			
5	5	72			
6	6	111			

Step 2 **a.** Make another spreadsheet and repeat Step 1 for the cubic polynomial function with equation $y = x^3 - 3x^2 + 4x - 5$.

b. Define a fifth column that finds the difference between consecutive terms of the fourth column.

Step 3 Make a conjecture about what will happen if you calculate four consecutive sets of differences of y-values with $y = -2x^4 + 8x^3 + 11x^2 - 3x$ for $x = 1$ to 7.

The Polynomial-Difference Theorem

Activity 1 shows that if you evaluate a polynomial of degree n for consecutive integer values of x and take differences between consecutive y-values, then after n sets of differences you get a constant difference. You can see the results of the calculations for the polynomial of Step 3 at the right.

x	1	2	3	4	5	6	7
$y = -2x^4 + 8x^3 + 11x^2 - 3x$	14	70	144	164	10	-486	-1540

56 74 20 −154 −496 −1054

18 −54 −174 −342 −558

−72 −120 −168 −216

4th differences are equal. −48 −48 −48

Consider again the linear function $y = 2x + 7$. Instead of using consecutive integers for x-values, use the arithmetic sequence -5, -1, 3, 7, 11, ...

x	-5	-1	3	7	11
$y = 2x + 7$	-3	5	13	21	29

1st differences 8 8 8 8

Again the 1st differences are equal. Each of these examples is an instance of the following theorem.

Polynomial-Difference Theorem

$y = f(x)$ is a polynomial function of degree n if and only if, for any set of x-values that form an arithmetic sequence, the nth differences of corresponding y-values are equal and the $(n - 1)$st differences are not equal.

The Polynomial-Difference Theorem provides a technique to determine whether a polynomial function of a particular degree can be an exact model for a set of points. The technique is called the **method of finite differences**. From a table of *y*-values corresponding to an arithmetic sequence of *x*-values, take differences of consecutive *y*-values and continue to take differences of the resulting *y*-value differences as needed. Only if those differences are eventually constant is the function a polynomial function, and the number of the differences indicates the polynomial function's degree.

Example 1

Consider the data from the beginning of the lesson. Use the method of finite differences to determine the degree of the polynomial function mapping *s* onto *d*.

Solution Notice that the values of the independent variable *s* form an arithmetic sequence, so the Polynomial-Difference Theorem applies. Calculate the differences.

s	0	10	20	30	40
d	0	8	33	75	134

1st differences are not equal. 8 25 42 59

2nd differences are equal. 17 17 17

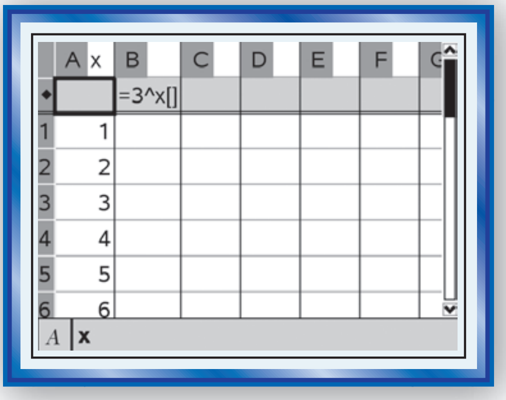
d is a 2nd-degree polynomial function of s because the 2nd differences are equal.

The Polynomial-Difference Theorem does *not* generalize to nonpolynomial functions.

Activity 2

Step 1 Make a spreadsheet to show *x*- and *y*-values of $y = 3^x$ for $x = 1$ to 7.

Step 2 Analyze the data using the method of finite differences through at least 3rd differences. Describe what you observe.



Activity 2 shows that when the method of finite differences is used with functions other than polynomial functions, the differences do not become constant.

GUIDED

Example 2

Consider the sequence a defined by the recursive formula

$$\begin{cases} a_1 = 3 \\ a_n = 2a_{n-1} + 1, \text{ for integers } n \geq 2. \end{cases}$$

a. **Identify the first seven terms of this sequence.**

b. **Use the method of finite difference to decide if there is an explicit polynomial formula for this sequence.**

Solution

a. From the recursive definition, the sequence is 3, 7, ___?___, ___?___, ___?___, ___?___, ...

b. Take differences between consecutive terms.

The pattern of differences appears to repeat and (will/will not) eventually give constant differences. So, there (is/is not) an explicit ___?___ formula for this sequence.

a_n	3	7	?	?	?	?	
1st differences	4	8	?	?	?		
2nd differences	4	8	?	?			

Questions

COVERING THE IDEAS

In 1 and 2, refer to Activity 1.

1. How many sets of differences did you need to take for the cubic function before the values were equal?

2. How many sets of differences would you need to take for the function with equation $y = 2x^2 + 3x^5 + x$?

3. If a set of x-values is an arithmetic sequence, the 11th differences of corresponding y-values are not all equal, but the 12th differences are equal, is y a polynomial function of x? If so, what is its degree? If not, why not?

4. a. According to the Polynomial-Difference Theorem, how many sets of differences will it take to get equal differences when $y = 5x^7 - 4x^5 + 6x^2$?

   b. Check your answer to Part a by making a spreadsheet to calculate y and the finite differences for $x = -2, -1, 0, 1, 2, 3, 4,$ and 5.

5. Tanner calculated three sets of finite differences for $y = x^2$ beginning with the table of values at the right. While Tanner knows this is a polynomial function, he did not get a constant set of differences. Explain why.

x	–3	0	1	4	5	7
y	9	0	1	16	25	49

6. Consider the sequence $\begin{cases} a_1 = 0.3 \\ a_n = a_{n-1} - 1 \text{ for integers } n \geq 2 \end{cases}$.

   a. Generate the first seven terms of the given sequence.

   b. Tell whether the sequence can be described explicitly by a polynomial function of degree less than 4.

In 7 and 8, use the data points listed in the table.

a. Determine if y is a polynomial function of x of degree ≤ 5.

b. If the function is a polynomial, state its degree.

7.

x	–3	–2	–1	0	1	2	3	4	5
y	67.5	16	1.5	0	–0.5	0	13.5	64	187.5

8.

x	0	1	2	3	4	5	6	7	8	9
y	1	1	0	–1	0	7	28	79	192	431

APPLYING THE MATHEMATICS

9. Consider the following sequence of sums of products of three consecutive integers.

   $f(1) = 1 \cdot 2 \cdot 3 = 6$

   $f(2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 = 30$

   $f(3) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 = 90$

   $f(4) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 = 210$

   $f(5) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 = 420$

   $f(6) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + 6 \cdot 7 \cdot 8 = 756$

   $f(7) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + 6 \cdot 7 \cdot 8 + 7 \cdot 8 \cdot 9 = 1260$

   Determine whether or not there is a polynomial function f of degree less than 5 that models these data exactly.

10. Let $f(n)$ = the sum of the 4th powers of the integers from 1 to n.

   $f(1) = 1^4 = 1$

   $f(2) = 1^4 + 2^4 = 17$

   $f(3) = 1^4 + 2^4 + 3^4 = 98$

   $f(4) = 1^4 + 2^4 + 3^4 + 4^4 = 354$, and so on.

   a. Find $f(5)$, $f(6)$, $f(7)$, and $f(8)$.

   b. According to the Polynomial-Difference Theorem, what is the degree of the polynomial $f(n)$?

11. a. Fill in the table of values for the linear function with equation $y = mx + b$.

   b. Find the first differences for the table in Part a and explain your results.

x	0	1	2	3	4
y	?	?	?	?	?

12. If the second differences for $y = kx^2$ all equal $\frac{2}{9}$, what is the value of k?

REVIEW

13. Consider the function f with equation $f(x) = 3x^4 - 10x^2 - 8x + 10$. **(Lessons 11-6, 11-5)**

   a. How many roots does $f(x)$ have? Justify your answer.

   b. According to the Rational-Root Theorem, what are the possible rational roots?

   c. Find all roots of $f(x)$. Write all rational roots as fractions. Approximate all irrational roots to the nearest hundredth.

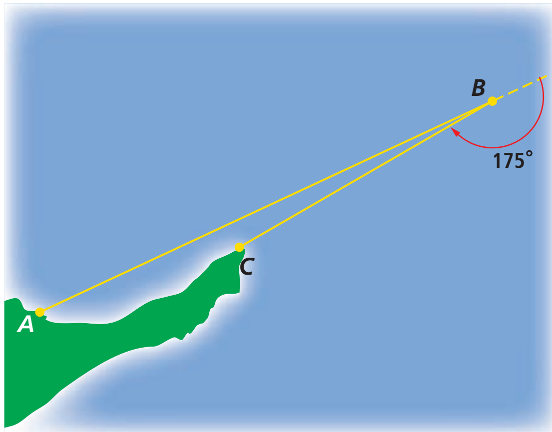
14. Find all the roots of $z^4 - 1 = 0$ by factoring and using the Zero-Product Theorem. **(Lessons 11-6, 11-3)**

15. Let $P(x) = (x - 1)(-x^2 + 3x + 3)$.

   a. Rewrite $P(x)$ in standard form.

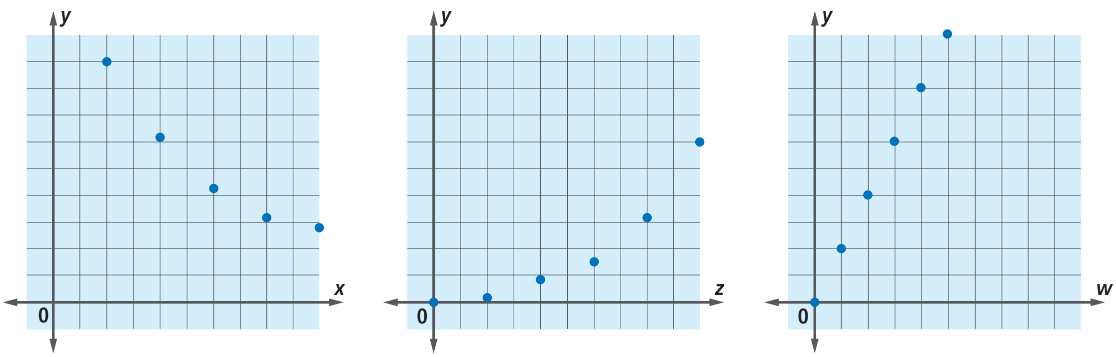
   b. How many x-intercepts does the graph of $y = P(x)$ have? Find the exact value of the largest of these. **(Lessons 11-4, 11-1, 6-7)**

16. Refer to the figure at the right. A boat sails 20 miles from A to B, then turns 175° as indicated and sails 12 miles to point C. How far is A from C? **(Lesson 10-8, Previous Course)**



17. Solve the system $\begin{cases} 4x - 2y + 3z = 1 \\ 8x - 3y + 5z = 4 \\ 7x - 2y + 4z = 5 \end{cases}$ using matrices. **(Lesson 5-6)**

18. **Multiple Choice** Determine which equation describes the relationships graphed below, where k is a constant. **(Lesson 2-8)**



z, *w* constant   *w*, *x* constant   *x*, *z* constant

A $y = \frac{kwz}{x^2}$ B $y = kwxz$ C $y = \frac{kwz^2}{x}$ D $y = \frac{kwx^2}{z}$

EXPLORATION

19. Find a sequence for which the 3rd differences are all 30.