Lesson

11-6 Solving All Polynomial Equations

Vocabulary

double root, root of multiplicity 2

multiplicity

BIG IDEA Every polynomial equation of degree $n \geq 1$ has exactly n solutions, counting multiplicities.

What Types of Numbers Are Needed to Solve Polynomial Equations?

An exact solution to any linear equation with real coefficients is always a real number, but exact solutions to quadratic equations with real coefficients sometimes are not real. It is natural to wonder whether exact solutions to higher-order polynomial equations require new types of numbers beyond the complex numbers.

As early as 2000 BCE, the Babylonians developed algorithms to solve problems using quadratic polynomials. The study of polynomials progressed to cubics and quartics, with much early work done in Persia, China, and Italy. In 1535, Niccolo Tartaglia discovered a method for solving all cubic equations. Soon after that, Ludovico Ferrari discovered how to solve any quartic equation.

Surprisingly, no numbers beyond complex numbers are needed to solve linear, quadratic, cubic, and quartic polynomial equations. However, when over 250 years passed without finding a general solution to every ***quintic*** (5th-degree) polynomial equation, mathematicians began wondering whether new numbers might be needed. At last, in 1824, the Norwegian mathematician Niels Henrik Abel proved that there is no general formula for solving quintic polynomial equations. Shortly after that, Evariste Galois deduced that there is no general formula involving complex numbers for finding roots of all polynomials of degree five or higher.

But how do we know that new numbers beyond the complex numbers were not necessary? In 1797, at the age of 18, Carl Gauss offered a proof of the following theorem, the name of which indicates its significance.

Fundamental Theorem of Algebra

Every polynomial equation $P(x) = 0$ of any degree ≥ 1 with complex number coefficients has at least one complex number solution.

Mental Math

How many lines of symmetry does each of the following have at a minimum?

a. a rhombus

b. a rectangle

c. an isosceles triangle

d. a circle

Carl Friedreich Gauss is often called "the prince of mathematicians."

From the Fundamental Theorem of Algebra and the Factor Theorem it is possible to prove that *every solution to a polynomial equation with complex coefficients is a complex number.* (Remember that complex numbers include the real numbers.) Thus, no new type of number is needed to solve higher-degree polynomial equations. So, for instance, the solutions to $x^5 + 2x^4 - 3ix^2 + 3 + 7i = 0$ are complex numbers.

How Many Complex Solutions Does a Given Polynomial Equation Have?

It is easy to show that every solution to a polynomial equation is complex when the degree of the polynomial is small. For example, if $a \neq 0$, the linear equation $ax + b = 0$ has one root: $x = -\frac{b}{a}$. Therefore, every linear polynomial has exactly one complex root.

The quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, has two solutions, given by the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. When the discriminant $b^2 - 4ac > 0$, the roots are real numbers. When $b^2 - 4ac < 0$, the roots are nonreal complex numbers. When the discriminant $b^2 - 4ac = 0$, the two roots are equal, and the root is called a **double root**, or a **root of multiplicity 2**. For instance, consider $x^2 - 10x + 25 = 0$. Here $b^2 - 4ac = (-10)^2 - 4 \cdot 1 \cdot 25 = 0$, so there is a double root. Because $x^2 - 10x + 25 = (x - 5)^2$, that double root is 5. So every quadratic has two complex roots, although the two roots might not be distinct.

In general, the **multiplicity** of a root r in an equation $P(x) = 0$ is the highest power of the factor $x - r$. For example, in $17(x - 4)(x - 11)^3 = 0$, 11 is a root of multiplicity 3, and 4 is a root of multiplicity 1. So $P(x) = 0$ has four roots altogether.

STOP **QY1**

> **QY1**
>
> State the multiplicity of each root of $4(x - 3)^2 \cdot (x + 5)^4(x - 17) = 0$. How many roots does the equation have altogether?

Activity

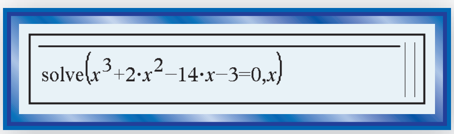
Consider the polynomial function P defined by $P(x) = x^3 + 2x^2 - 14x - 3$.

Step 1 Verify that 3 is a zero of P. What theorem justifies the conclusion that $x - 3$ is a factor of $P(x)$?

Step 2 If $P(x) = (x - 3)Q(x)$, what is the degree of $Q(x)$?

Step 3 Without finding $Q(x)$ explicitly, how many roots does the equation $Q(x) = 0$ have? Why are these roots also roots of $P(x) = 0$?

Step 4 Check your answer to Step 3 by solving $P(x) = 0$ on a CAS.



The Fundamental Theorem of Algebra only guarantees the existence of a single complex (possibly real) root r_1 for any polynomial $P(x)$. However, using the Factor Theorem, you can rewrite $P(x) = (x - r_1) \cdot Q(x)$, where the degree of $Q(x)$ is one less than the degree of $P(x)$. By applying the Fundamental Theorem of Algebra to $Q(x)$, you get another root r_2, and therefore another factor. So you could write $P(x) = (x - r_1)(x - r_2)Q_2(x)$, where the degree of $Q_2(x)$ is *two* less than the degree of $P(x)$. This process can continue until you "run out of degrees," that is, until $Q(x)$ is linear.

For example, if you start with a 4th degree polynomial $P(x)$, you can factor it as

$$\begin{aligned} P(x) &= (x - r_1)Q_1(x) \\ &= (x - r_1)(x - r_2)Q_2(x) \\ &= (x - r_1)(x - r_2)(x - r_3)Q_3(x), \end{aligned}$$

and now $Q_3(x)$ is linear. So, $P(x) = 0$ has four roots: r_1, r_2, r_3, and the single root of $Q_3(x) = 0$. These four roots are *not* necessarily four different real numbers. In fact, some or all of them may be nonreal complex numbers. This conclusion is summarized in the following theorem.

Number of Roots of a Polynomial Equation Theorem

Every polynomial equation of degree n has exactly n roots, provided that multiple roots are counted according to their multiplicities.

STOP QY2

QY2

How many roots does each equation have?

a. $5x^{12} - 64x^3 + 4x^2 + 1 = 0$

b. $4ex^5 + 3ix^2 - (2 + i)x + 4 = 0$

Finding Real Solutions to Polynomial Equations

The Number of Roots of a Polynomial Equation Theorem tells you how many roots a polynomial equation $P(x) = k$ has. It does not tell you how to find the roots, nor does it tell you how many of the roots are real. To answer these questions, you can apply the methods studied in this chapter for finding and analyzing zeros of polynomial functions.

Example

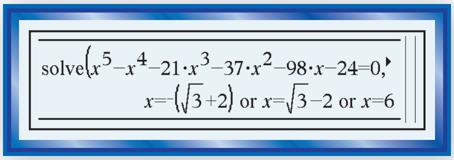
Consider the polynomial function P defined by $P(x) = x^5 - x^4 - 21x^3 - 37x^2 - 98x - 24$.

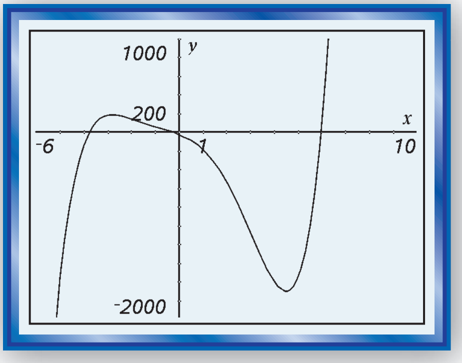
a. How many real zeros does P have?

b. How many nonreal complex zeros does P have?

Solution 1

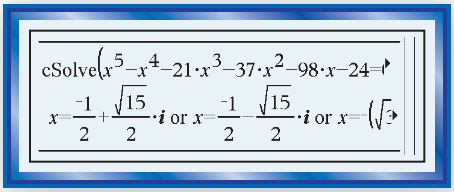
a. Solve $P(x) = 0$ on a CAS in real-number mode. The CAS shows that P has three real zeros. One zero, 6, is rational, but the other two real zeros, $-2 \pm \sqrt{3}$, are irrational.





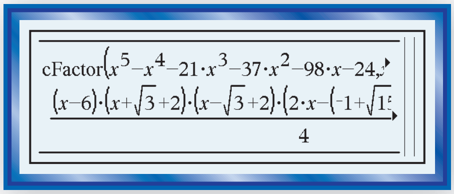
Confirm with a graph of $y = P(x)$ that the three x-intercepts correspond to the three real zeros.

b. Because the degree of $P(x)$ is five, there are five zeros altogether by the Number of Roots of a Polynomial Equation Theorem. Since there are three real zeros, there are two nonreal complex zeros. Solve in complex-number mode on a CAS to find all five real and nonreal zeros. There are two nonreal zeros, $-\frac{1}{2} \pm \frac{\sqrt{15}}{2}i$. So, of the five zeros of P(x), one is rational, two are irrational, and two are nonreal complex.



Check Factor $P(x)$ on a CAS in complex mode. Part of the solution line is shown at the right. The entire output is

$$\frac{(x - 6)(x + \sqrt{3} + 2)(x - \sqrt{3} + 2)(2x - (-1 + \sqrt{15}\,i))(2x + 1 + \sqrt{15}\,i)}{4}.$$



When the constant $\frac{1}{4}$ is factored out, there are five factors that can be set equal to zero and solved for x. Solving shows that one root of P(x) is rational, two are irrational, and two are nonreal. It checks.

Questions

COVERING THE IDEAS

1. State the Fundamental Theorem of Algebra.

2. Zelda is confused. The polynomial $x^3 - 25x$ is supposed to have three complex roots, but the roots 0, 5, and –5 are all real numbers. Resolve Zelda's confusion.

3. **Multiple Choice** The equation $\pi w^3 - 3iw + \frac{1}{17} = 0$ has how many complex solutions?

   A none **B** two **C** three **D** four

4. Who first proved the Fundamental Theorem of Algebra?

In 5 and 6, $a \neq 0$. Solve for *x*.

5. $ax + b = 0$

6. $ax^2 + bx + c = 0$

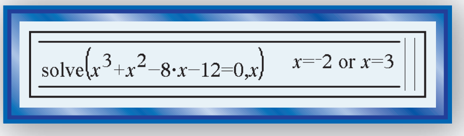
In 7–9, an equation is given. Find and classify all solutions as rational, irrational, or nonreal. Then, identify any multiple roots.

7. $x^2 + 16x + 64 = 0$

8. $y^3 - 8y = 0$

9. $(z - 3)^4(7z - 22)(z^2 - 6z + 25) = 0$

10. a. The equation $x^3 + x^2 - 8x - 12 = 0$ has only two roots, as the CAS screen shows at the right. However, the Number of Roots of a Polynomial Equation Theorem says that a cubic equation has three roots. How is this possible?

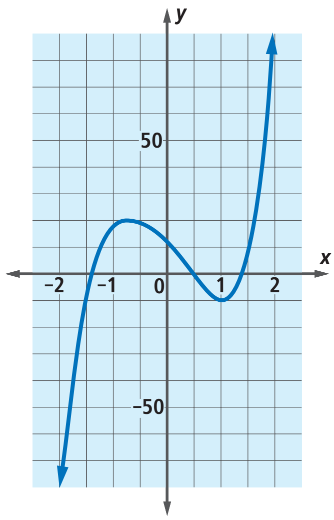


b. Without using a CAS, give a factorization of $x^3 + x^2 - 8x - 12$ that explains the result in Part a.

11. How many complex roots does the equation $x^5 + 14ix = 0$ have?

APPLYING THE MATHEMATICS

12. Consider the polynomial function *P* defined by $P(x) = 4x^5 + x^3 - 5x^2 - 18x + 10$ graphed at the right.



a. How many real zeros does *P* have?

b. How many nonreal zeros does it have?

c. The graph shows that the real zeros are approximately -1.4, $\frac{1}{2}$, and 1.4. Which of these is an exact zero?

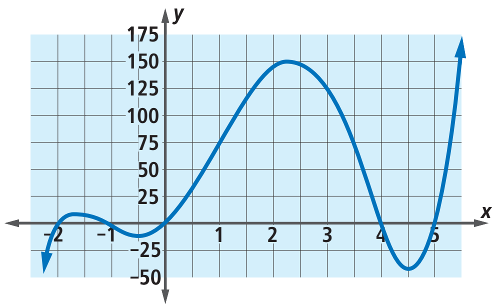
d. How many irrational zeros does this function have?

In 13 and 14, solve the equation.

13. $-4ix + 8 + 16i = 0$

14. $ix^2 + 10x + 11i = 0$

15. Consider $f(x) = x^5 - 6x^4 - 5x^3 + 42x^2 + 40x + k$, where *k* is a real number. The case where $k = 0$ is graphed at the right.



a. What effect does changing the value of *k* have on the graph of $y = f(x)$?

b. Find a value of *k* for which *f* has exactly two nonreal zeros.

c. For what values of *k* (approximately) does *f* have exactly four nonreal zeros?

16. The 4th roots of -16 are nonreal complex numbers. If *z* is a 4th root of -16, then $z^4 = -16$ and $z^4 + 16 = 0$.

a. How many complex solutions does $z^4 + 16 = 0$ have?

b. Use a CAS to find all of the complex fourth roots of -16.

c. By similar logic as in Parts a and b, -8 has __?__ complex cube roots. Find them.

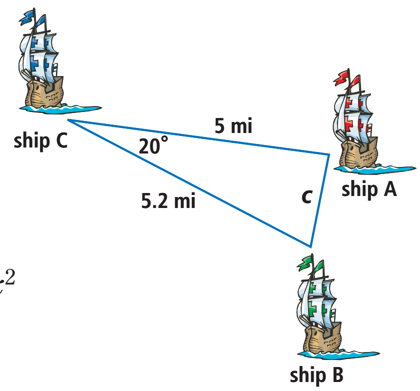
REVIEW

17. A 3rd-degree polynomial with leading coefficient 3 has roots -4, $\frac{2}{3}$, and 5. Find an equation for the polynomial. **(Lesson 11-4)**

18. The sum of the cube and the square of a number is 3. To the nearest thousandth, what is the number? **(Lessons 11-3, 11-1)**

19. Myron makes a bread pan by cutting squares of side length w from each corner of a 5-inch by 10-inch sheet of aluminum and folding up the sides. Let $V(w)$ represent the volume of the pan. **(Lesson 11-2)**
    - **a.** Find a polynomial formula for $V(w)$.
    - **b.** Can Myron's process yield a pan with volume of 25 in^3?
    - **c.** What value of w produces the pan with the greatest volume?

20. **a.** Which is more help in finding the distance c in the diagram at the right, the Law of Sines or the Law of Cosines?
    - **b.** Find c. **(Lesson 10-8)**



In 21 and 22, solve. (Lessons 10-6, 9-5)

21. $2 \log x = 4$

22. $2 \sin x = 4$

23. **a.** State an equation for the image of the graph of $y = 3x^2$ under $T_{-3,4}$.
    - **b.** Graph the image in Part a. **(Lesson 6-3)**

24. The wind force on a vertical surface varies jointly as the area A of the surface and the square of the wind speed S. The force is 340 newtons on a vertical surface of 1 m^2 when the wind blows at $18\frac{\text{m}}{\text{sec}}$. Find the force exerted by a $35\frac{\text{m}}{\text{sec}}$ wind on a vertical surface of area 2 m^2. (*Note:* One newton equals one $\frac{\text{kilogram-meter}}{\text{sec}^2}$.) **(Lesson 2-9)**

25. Make a spreadsheet to create the table of values for $y = 4x^2 - 13x$ as shown at the right. Use the table to find y when $x = 3.7$. **(Lesson 1-6)**

	A	B	C	D	E
◆		=4*a[]^2−13*a[]			
1	-6	222			
2	-5.9	215.94			
3	-5.8	209.96			
4	-5.7	204.06			
5	-5.6	198.24			
6	-5.5	192.5			

EXPLORATION

26. **Fill in the Blanks** Because $1^n = 1$ for all n, $x^n = 1$ has the solution 1 for all n. Thus, for all n, by the Factor Theorem, $x^n - 1$ has the factor $x - 1$.
    - **a.** $x^2 - 1$ is the product of $x - 1$ and ___?___.
    - **b.** $x^3 - 1$ is the product of $x - 1$ and ___?___.
    - **c.** $x^4 - 1$ is the product of $x - 1$ and ___?___.
    - **d.** Generalize Parts a–c.

QY ANSWERS

1. 3 is a root of multiplicity 2; –5 is a root of multiplicity 4; 17 is a root of multiplicity 1. It has 7 roots altogether.

2. a. 12

b. 5