Lesson

11-5 The Rational-Root Theorem

BIG IDEA The Rational-Root Theorem gives a criterion that any rational root of a polynomial equation must satisfy, and typically limits the number of rational numbers that need to be tested to a small number.

Remember that a *rational number* is a number that can be written as a quotient of two integers, that is, as a simple fraction. Any integer n is a rational number, because n can be written as $\frac{n}{1}$. Zero is a rational number because, for example, $0 = \frac{0}{26}$. The number 10.32 is rational because $10.32 = 10 + \frac{3}{10} + \frac{2}{100} = \frac{1032}{100}$. An *irrational number* is a real number that is not rational. The irrational numbers are infinite nonrepeating decimals, including $\sqrt{60}$, $\pi + 5$, and e. Nonreal complex numbers such as $3i$ and $i - 4$ are neither rational nor irrational.

Every *real* zero of a function f corresponds to an x-intercept of the graph of $y = f(x)$. But it can be difficult to tell from a graph which zeros are rational. However, there is a theorem that details the possible rational zeros that a polynomial function can have. The following Activity is about this theorem.

Activity

Let $Q(x) = (2x - 3)(9x + 4)$ and $P(x) = (2x - 3)(9x + 4)(5x + 7)$.

Step 1 Solve $Q(x) = 0$ and $P(x) = 0$ for x.

Step 2 Without expanding the polynomials, find the first and last terms of $Q(x)$ and $P(x)$ when they are written in standard form.

Step 3 Describe the connection between the denominators of the roots of the polynomial equation $Q(x) = 0$ and the coefficient of the first term of the expanded polynomial $Q(x)$. Repeat for $P(x)$.

Step 4 Describe the connection between the numerators of the roots of the polynomial equation $Q(x) = 0$ and the constant term of the expanded polynomial $Q(x)$. Repeat for $P(x)$.

Mental Math

a. Which is the better value, an 8-oz box of pasta for \$2.50 or a 12-oz box of pasta for \$3.50?

b. Should you buy a \$30 sweater using a 30%-off coupon or using a \$10-off coupon?

c. Do you save more when a \$25 instant rebate on contact lenses is taken before tax is calculated or after tax is calculated?

d. Which will pay more interest, an account with 3.5% interest compounded annually, or an account with continuously compounded interest with an APY of 3.5%?

A generalization of the results of the Activity is called the *Rational-Root Theorem,* or *Rational-Zero Theorem.*

Rational-Root (or Rational-Zero) Theorem

Suppose that all the coefficients of the polynomial function described by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

are integers with $a_n \neq 0$ and $a_0 \neq 0$. If $\frac{p}{q}$ is a root of $f(x)$ in lowest terms, then p is a factor of a_0 and q is a factor of a_n.

Stated another way, the Rational-Root Theorem says that if a simple fraction in lowest terms (a rational number) is a root of a polynomial function with integer coefficients, then the numerator of the rational root is a factor of the constant term of the polynomial, and the denominator of the rational root is a factor of the leading coefficient of the polynomial. A proof of this theorem is left to a later course.

Identifying Possible and Actual Rational Roots

Notice that the Rational-Root Theorem gives a way to decide the *possible* rational roots of a polynomial. It does not determine which of these possible roots are actual roots of the polynomial.

Example 1

Apply the Rational-Root Theorem to *identify* possible rational roots of $f(x)$ when $f(x) = 4x^4 + 3x^3 + 4x^2 + 11x + 6$.

Solution Let $\frac{p}{q}$ in lowest terms be a rational root of $f(x)$. Then p is a factor of 6 and q is a factor of 4.

So, p equals ±1, ±2, ±3, or ±6, and q equals ±1, ±2, or ±4. Now take all possible quotients $\frac{p}{q}$. It looks like there are as many as $8 \cdot 6 = 48$ possible quotients, but actually there are fewer because many of them, such as $\frac{6}{2}$ and $\frac{-3}{-1}$, are equal. So, the possible rational roots are ±1, ±2, ±3, ±6, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{1}{4}$, and $\pm\frac{3}{4}$.

STOP QY1

Example 1 shows that there can be several possible rational roots for a polynomial. You can test each possible root by hand, but with a graphing utility, a CAS, the Rational-Root Theorem, and the Factor Theorem, you can greatly reduce the time it takes to identify roots.

QY1

$\frac{2}{4}$ and $\frac{6}{2}$ are both possible rational roots of $f(x)$ in Example 1. Why do they not appear in this form on the list in the solution to Example 1?

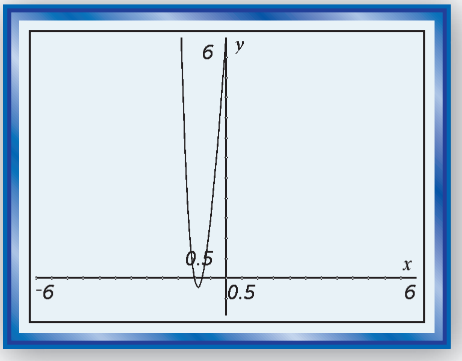
Example 2

Use the Rational-Root Theorem to *find* all rational roots of $f(x) = 4x^4 + 3x^3 + 4x^2 + 11x + 6$ from Example 1.

Solution The possible rational roots of $f(x)$ are $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}$, and $\pm\frac{3}{4}$.

The smallest possible rational root is –6 and the largest is 6, so graph f on the interval $-6 \leq x \leq 6$ to see where rational roots might be.

The graph at the right shows that there are two real roots on the interval $-1 \leq x \leq 0$.



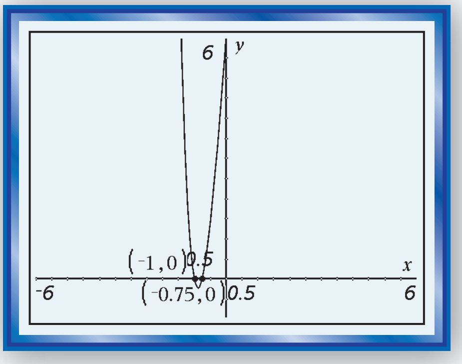
The possible rational roots for $f(x)$ in that interval are $-\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}$, and –1. Test these values in the equation for f.

$f\left(-\frac{1}{4}\right) \approx 3.47$ $\qquad f\left(-\frac{1}{2}\right) \approx 1.38$

$f\left(-\frac{3}{4}\right) = 0$ $\qquad f(-1) = 0$

So, –1 and $-\frac{3}{4}$ are the only rational roots of $f(x)$.

Check Use a graphing utility to find the x-intercepts of the graph of f. The answers check.



Irrational Roots

The Rational-Root Theorem identifies all possible rational roots of a polynomial with integer coefficients. So, any real roots that are not identified must be irrational.

Example 3

In a Hans Magnus Enzensberger book, a boy named Robert argues with a sprite about $\sqrt{2}$. The sprite tries to convince Robert that $\sqrt{2}$ is irrational by showing him that its decimal expansion is infinite. How could the sprite show that $\sqrt{2}$ is irrational using the Rational-Root Theorem?

Solution $\sqrt{2}$ is a solution to the equation $x^2 = 2$ and a root of $x^2 - 2 = 0$. By the Rational-Root Theorem, if $\frac{a}{b}$ is a rational root of $x^2 - 2 = 0$, then a is a factor of 2 and b is a factor of 1.

Thus, the only possible rational roots of $x^2 - 2 = 0$ are $\pm\frac{2}{1}$ and $\pm\frac{1}{1}$, that is, 2, -2, 1, and -1.

Substitute to see if any of these numbers is a root of the equation $x^2 - 2 = 0$.

$1^2 - 2 = -1 \neq 0 \quad (-1)^2 - 2 = -1 \neq 0$

$2^2 - 2 = 2 \neq 0 \quad (-2)^2 - 2 = 2 \neq 0$

None of the possible rational roots is a root of the equation. Thus, $x^2 - 2 = 0$ has no rational roots, so $\sqrt{2}$ must be irrational.

STOP **QY2**

QY2

What equation can you consider in order to prove that $\sqrt[3]{9}$ is irrational?

Questions

COVERING THE IDEAS

1. Suppose that $\frac{7}{5}$ is a root of a polynomial equation with integer coefficients. What can you say about the leading coefficient and the constant term of the polynomial?

2. Does the Rational-Root Theorem apply to finding roots of $Q(x) = 10x^4 - 4\sqrt{2}x + 9$? Explain your response.

In 3 and 4, a polynomial equation is given.

a. Use the Rational-Root Theorem to list the possible rational roots.

b. Find all of the rational roots.

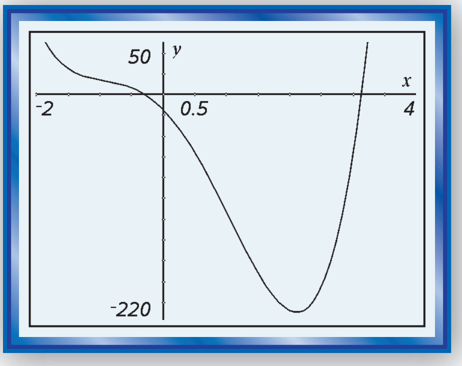
3. $8x^3 - 4x^2 + 44x + 24 = 0$

4. $f(n) = -n^2 + 3n^3 + 5n^5 - 8n + 12 - 11n^4$

5. The graph of a polynomial function with equation $R(x) = 7x^4 - 2x^3 - 42x^2 - 61x - 14$ is shown at the right.

   a. Using the Rational-Zero Theorem, list all possible rational zeros of R.

   b. Which possible rational zeros of R are actual zeros? How did you decide which roots to test?



6. Prove that $\sqrt[3]{9}$ is irrational.

APPLYING THE MATHEMATICS

7. Consider the function f, where $f(n) = 2n^4 + 5n^3 + 12$.

   a. List all possible rational zeros of this function.

   b. Graph f and explain why it has no rational zeros.

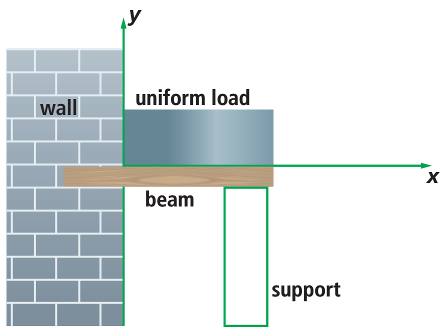
8. Explain how the Rational-Root Theorem and the Factor Theorem can be used to factor a polynomial equation with only rational roots.

9. When a polynomial equation with integer coefficients has a root of the form $a + \sqrt{p}$, where p is not a perfect square, then $a - \sqrt{p}$ must also be a root of the equation.
   a. Use this fact to find a polynomial equation with integer coefficients in which $a + \sqrt{p}$ is a root.
   b. If p is a prime number and a is an integer, show that $a + \sqrt{p}$ is irrational.

10. Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $g(x) = 7a_3x^3 + 7a_2x^2 + 7a_1x + 7a_0$, where a_3, a_2, a_1, and a_0 are integers with $a_3 \neq 0$ and $a_0 \neq 0$.
   a. How are the possible rational zeros of these functions related? Explain your reasoning.
   b. Let $f(x)$ be defined as in Part a and $h(x) = k \cdot f(x)$, where k is a nonzero constant. How are the possible rational zeros of f and h related?

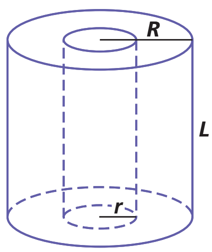
REVIEW

11. A horizontal beam has its left end built into a wall, and its right end resting on a support, as shown at the right. The beam is loaded with weight uniformly distributed along its length. As a result, the beam sags downward according to the equation $y = -x^4 + 24x^3 - 135x^2$, where x is the distance (in meters) from the wall to a point on the beam, and y is the distance (in hundredths of a millimeter) of the sag from the x-axis to the beam. **(Lesson 11-3)**



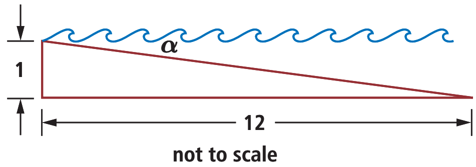
   a. What is the appropriate domain for x if the beam is 9 meters long?
   b. Find the zeros of this function.
   c. Tell what the roots represent in this situation.

12. Insulation tubing for hot water pipes is shaped like a cylindrical solid of outer radius R from which another cylindrical solid of inner radius r has been removed. The figure at the right shows a piece of insulation tubing. **(Lessons 11-1, 1-7, Previous Course)**



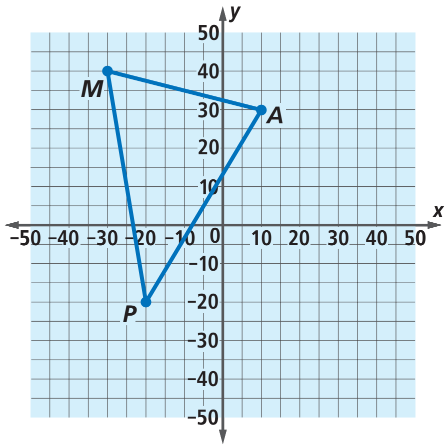
   a. Suppose the piece of tubing has length L. Find a formula for the volume V of the tube in terms of r, R, and L.
   b. Suppose $R = 1$ inch and $L = 6$ feet. Write a formula for V as a function of r.
   c. What is the degree of the polynomial in Part b?

13. The building code in one state specifies that accessibility ramps into public swimming pools must not drop more than one inch for every horizontal foot. What is the maximum angle of depression a ramp can make with the surface of the water? **(Lesson 10-2)**



14. Find the roots of $z^2 + 2z + 6 = 0$. **(Lesson 6-10)**

15. Triangle *MAP* is shown below. **(Lessons 4-7, 4-6)**



a. Graph the reflection image of $\triangle MAP$ over the x-axis. Label the vertices M', A', and P', respectively.

b. Graph the reflection image of $\triangle M'A'P'$ over the y-axis. Label the vertices M'', A'', and P'', respectively.

c. What single transformation maps $\triangle MAP$ onto $\triangle M''A''P''$?

EXPLORATION

16. Find information about *Descartes' Rule of Signs* for polynomials, and then explain what it tells you about $P(x)$ from the Activity in this lesson.

QY ANSWERS

1. $\frac{2}{4}$ and $\frac{6}{2}$ are not in lowest terms, which the theorem requires.

2. $x^3 - 9 = 0$