Lesson

11-4 The Factor Theorem

Vocabulary

zero of a polynomial, root of a polynomial

BIG IDEA If $P(x)$ is a polynomial, then a is a solution to the equation $P(x) = 0$ if and only if $x - a$ is a factor of $P(x)$.

Zeros of a Polynomial

Consider an equation in which there is a polynomial on each side, such as

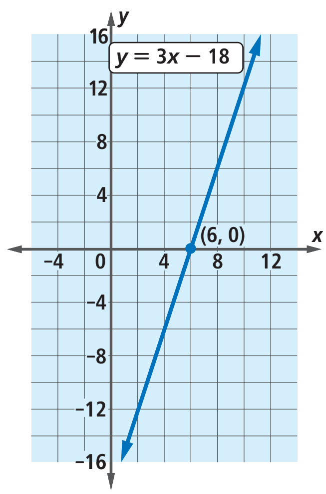
$$14x^3 + 3x - 10 = 5x - 9x^2 + 5.$$

Add the opposite of $5x - 9x^2 + 5$ to each side. Then

$$14x^3 + 9x^2 - 2x - 15 = 0.$$

In this way, every equality of two polynomials in x can be converted into an equivalent equation of the form $P(x) = 0$. More generally, any equation involving two expressions in x can be converted into an equation of the form $f(x) = 0$.

As you know, when f is a function, then a solution to the equation $f(x) = 0$ is called a *zero* of f. When P is a polynomial function, then a zero of P is also called a **zero** or **root of the polynomial** $P(x)$. For example, when $P(x) = 3x - 18$, then 6 is a zero, or root, of $P(x)$ because $3 \cdot 6 - 18 = 0$. A zero of $P(x)$ is an x-intercept of the graph of P, as shown at the right.



The factors of a polynomial $P(x)$ are connected to its zeros, or x-intercepts. Understanding this connection can help you to solve equations with polynomials and to understand their algebraic structure.

Recall that a product of numbers equals 0 if and only if one of the factors equals 0. We call this result the *Zero-Product Theorem*.

Zero-Product Theorem

For all a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

For instance, if $f(x) = 0$ and $f(x) = g(x) \cdot h(x)$, then $g(x) = 0$ or $h(x) = 0$. This is one reason for manipulating an equation so that 0 is on one side.

Mental Math

Let h be a sequence with explicit formula $h_n = h_0 \cdot r^n$ that models the maximum height of a ball after n bounces. Are the following *always*, *sometimes but not always*, or *never* true?

a. r is greater than 1.

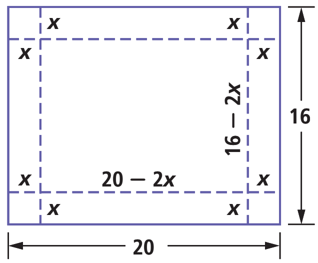
b. r is less than zero.

c. h_0 is greater than h_3.

d. h_4 is greater than h_3.

Example 1

In Example 4 of Lesson 11-2, the volume $V(x)$ of the box shown at the right is given by $V(x) = x(20 - 2x)(16 - 2x) = 4x^3 - 72x^2 + 320x$. Find the zeros of V.



Solution To find the zeros, solve $x(20 - 2x)(16 - 2x) = 0$. By the Zero-Product Theorem, at least one of these factors must equal zero.

Either $x = 0$ or $20 - 2x = 0$ or $16 - 2x = 0$.
So, $x = 0$ or $x = 10$ or $x = 8$.

The zeros are 0, 8, and 10.

Check Notice that the zeros are the three values of x for which one of the dimensions of the box is 0, so there would be no volume.

Activity 1

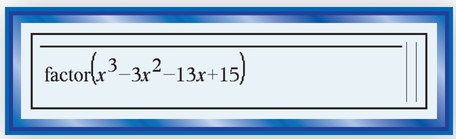
MATERIALS CAS

Work with a partner and use a graphing utility. One partner should work with $p(x) = x^3 - 3x^2 - 13x + 15$, the other with $q(x) = x^4 - x^3 - 24x^2 + 4x + 80$.

Step 1 Graph your polynomial equation in the window $-10 \le x \le 10$; $-100 \le y \le 100$.

Step 2 Find the x-intercepts of your graph.

Step 3 Factor your polynomial on a CAS.



Step 4 Compare your results with your partner's. How are the x-intercepts of each graph related to the factors of each polynomial?

Step 5 Use your answer to Step 4 to write a polynomial whose graph has x-intercepts 4, 5, and –1.

The results of Activity 1 show that the zeros of a polynomial function correspond to the polynomial's linear factors. For example, the zeros of p are –3, 1, and 5, and the factors of $p(x)$ are $x + 3$, $x - 1$, and $x - 5$. The following theorem generalizes this relationship.

Factor Theorem

$x - r$ is a factor of $P(x)$ if and only if $P(r) = 0$, that is, r is a zero of P.

Proof If $x - r$ is a factor of $P(x)$, then for all x, $P(x) = (x - r) \cdot Q(x)$, where $Q(x)$ is some polynomial. Substitute r for x in this formula.

$$P(r) = (r - r) \cdot Q(r) = 0 \cdot Q(r) = 0.$$

Proving the other direction of the theorem, that $P(r) = 0$ implies that $x - r$ is a factor of $P(x)$, requires more work. Consider the special case where $r = 0$. Write $P(x)$ in general form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Then $P(r) = P(0) = a_n 0^n + a_{n-1}0^{n-1} + \ldots + a_1 0 + a_0$.

So $P(0) = a_0$, the constant term.

Thus, when $r = 0$ and $P(0) = 0$, $a_0 = 0$.

So, x is a factor of every term of $P(x)$, and we can write

$$P(x) = x(a_n x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_1).$$

So $x = x - 0$ is a factor of $P(x)$.

If $r \neq 0$ and $P(r) = 0$, then the graph of $y = P(x)$ contains the point $(r, 0)$. Think of the graph of $y = P(x)$ as a translation image r units to the right of the graph of a polynomial function G that contains $(0, 0)$. Because $G(0) = 0$, the case above applies to $G(x)$, so $G(x) = x \cdot H(x)$ for some polynomial $H(x)$. By the Graph-Translation Theorem, $P(x)$ can be formed by replacing x in $G(x)$ by $x - r$. Therefore, $P(x) = (x - r) \cdot H(x - r)$, and so $x - r$ is a factor of $P(x)$.

STOP **QY1**

> **QY1**
>
> 5 is a zero of q in Activity 1. What factor of $q(x)$ is associated with this zero?

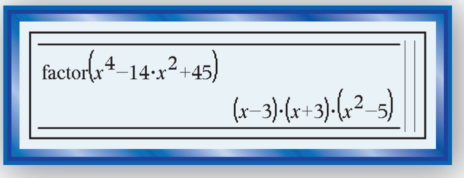
Finding Zeros by Factoring

Both the Zero-Product Theorem and the Factor Theorem provide methods for finding zeros of polynomials by factoring.

Example 2

Find the roots of $P(x) = x^4 - 14x^2 + 45$ by factoring.

Solution 1 Use a CAS command to factor $P(x)$ over the rationals, as shown at the right.



The roots are the solutions to $(x - 3)(x + 3)(x^2 - 5) = 0$.

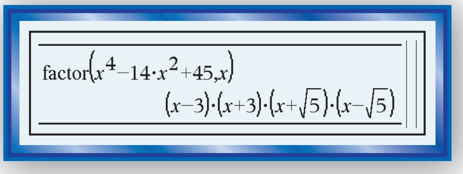
Use the Zero-Product Theorem.

$$\begin{array}{rcccccc} x - 3 = 0 & \text{or} & x + 3 = 0 & \text{or} & x^2 - 5 = 0 \\ x = 3 & \text{or} & x = -3 & \text{or} & x = \sqrt{5} \text{ or } -\sqrt{5} \end{array}$$

So, the roots of $P(x)$ are 3, -3, $\sqrt{5}$ and $-\sqrt{5}$.

Solution 2 Use a CAS command to factor $P(x)$ over the reals. Notice the difference in the command in the screenshot at the right. This shows

$$P(x) = (x - 3)(x + 3)(x + \sqrt{5})(x - \sqrt{5}).$$



Now use the Factor Theorem.

The factor $x - 3$ means that 3 is a root of P.
The factor $x + 3 = x - (-3)$ means that -3 is a root of P.
The factor $x - \sqrt{5}$ means that $\sqrt{5}$ is a root of P.
The factor $x + \sqrt{5} = x - (-\sqrt{5})$ means that $-\sqrt{5}$ is a root of P.

So, the roots of P(x) are 3, -3, $\sqrt{5}$ and $-\sqrt{5}$.

STOP **QY2**

QY2

Use the factorization of $x^2 - 225$ to find the roots of $f(x) = x^2 - 225$.

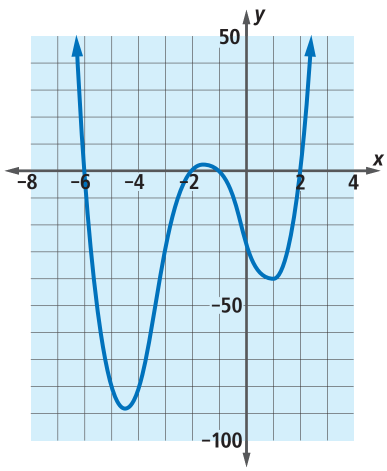
Finding Equations from Zeros

The Factor Theorem also says that if you know the zeros of a polynomial function, then you can determine the polynomial's factors.

GUIDED

Example 3

A polynomial function p with degree 4 and a leading coefficient of 1 is graphed at the right. Find the factors of $p(x)$ and use them to write a formula for $p(x)$.



Solution From the graph, the zeros appear to be -6, -2, -1, and 2.

By the Factor Theorem, the factors are $x - (-6)$, ___?___, ___?___, and $x - 2$. Therefore, $p(x) = (x + 6)($___?___$)($___?___$)(x - 2)$.

In standard form, $p(x) = x^4 + 7x^3 + 2x^2 - 28x - 24$.

Check Graph $y = p(x) = x^4 + 7x^3 + 2x^2 - 28x - 24$ in the window $-8 \le x \le 4$; $-100 \le y \le 50$. Does it match the given graph?

As you saw in Example 3, given a set of zeros, you can create a polynomial function with those zeros by multiplying the associated factors. However, if you do not restrict the degree or leading coefficient, different polynomial functions can have the same zeros.

Activity 2

Step 1 Find the zeros of each polynomial function below without graphing.

a. $f(x) = x(x + 4)(x - 3)$ **b.** $g(x) = 4x(x + 4)(x - 3)$

c. $h(x) = x^2(x + 4)(x - 3)$ **d.** $k(x) = -\frac{1}{4}x^2(x + 4)(x - 3)^3$

Step 2 Graph the four functions from Step 1 in the window $-5 \leq x \leq 5$; $-100 \leq y \leq 100$. How are their graphs similar? Describe any differences.

Step 3 Find an equation for a polynomial function whose zeros are 0, –4, and 3, but whose graph is different from the graphs in Step 2.

Activity 2 illustrates that a polynomial can be transformed in at least two ways to produce another polynomial with the same zeros:

- The original polynomial can be multiplied by a constant factor k.
- One or more factors of the original polynomial can be raised to a different positive integer power.

For example, every polynomial of the form $kx^a(x + 4)^b(x - 3)^c$ has the same zeros as the polynomial $x(x + 4)(x - 3)$.

Example 4

Find the general form of a polynomial function P whose only zeros are $-\frac{7}{2}$, $\frac{1}{4}$, and 2.

Solution By the Factor Theorem, $x - \left(-\frac{7}{2}\right)$, $x - \frac{1}{4}$, and $x - 2$ are factors of $P(x)$, and because P has no other zeros, these factors are the only factors. Any of these factors can be raised to any positive integer power, and the entire polynomial could be multiplied by any nonzero constant. So, $P(x) = k\left(x + \frac{7}{2}\right)^a\left(x - \frac{1}{4}\right)^b(x - 2)^c$ where $k \neq 0$ and a, b, c are positive integers.

From the given information in Example 4, you know the degree of $P(x)$ is at least three. However, you have no information about the exponents a, b, and c except that they are positive integers. Therefore, you cannot be sure of the degree of $P(x)$.

Questions

COVERING THE IDEAS

1. State the Zero-Product Theorem.

In 2 and 3, find the roots of the equation.

2. $(x - 5)(6x + 33) = 0$

3. $\left(-\frac{3}{5}k + 12\right)(k^2 - 2)(k + 1) = 0$

4. If $f(x) = 4x(x - 2)(x + 16)$, find the zeros of f.

5. **Multiple Choice** If the graph of a polynomial function intersects the x-axis at (5, 0) and (-2, 0), then which of the following must be two of the polynomial's linear factors?

A $x - 5$ and $x - 2$ B $x + 5$ and $x - 2$

C $x - 5$ and $x + 2$ D $x + 5$ and $x + 2$

6. Suppose that P is a polynomial function and $P(1.7) = 0$. According to the Factor Theorem, what can you conclude?

In 7 and 8, an equation for a polynomial function is given.

a. Factor the polynomial.

b. Find the zeros of the function.

7. $r(t) = 2t^3 - t^2 - 21t$

8. $w(b) = -b^4 + 4b^3 + 11b^2 - 30b$

9. a. Write the general form of an equation for a polynomial function whose only zeros are 5.6 and -2.9.

b. Graph two different third-degree polynomial functions satisfying the condition in Part a.

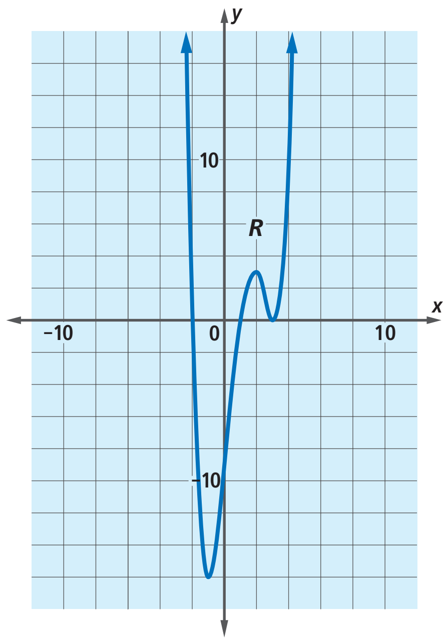
10. **Multiple Choice** A polynomial has real zeros 4, -2, and 6. What must the degree of the polynomial be?

A 2 B 3 C 8

D 12 E an integer ≥ 3

APPLYING THE MATHEMATICS

11. The graph of a polynomial function is shown at the right. There are no other x-intercepts and the degree of the function is 4. Give an equation for the function.



12. a. Is it possible to have a polynomial with integer coefficients that has 3 and $\frac{3}{5}$ as zeros, and no other zeros? Justify your answer.

b. Is it possible to have a polynomial with integer coefficients that has 3 and $\sqrt{5}$ as zeros, and no other real zeros? Justify your answer.

13. Consider the polynomial function $P(x) = x^3 + 5x^2 + 3x$.

a. Factor $P(x)$ over the rationals.

b. Use the factors you found in Part a to find the zeros of the function.

c. Your answer to Part b should suggest that there are other factors. Use a CAS to factor over the reals and compare your result to your factorization in Part a.

14. Find all possible values of a such that $x - 2$ is a factor of $x^3 + x^2 + ax + 4$.

15. The graph of the polynomial function with equation $y = \frac{1}{2}(x^2 + 1) \cdot (x + 2)^2(x - 3)$ has only two x-intercepts: -2 and 3. Why does the factor $x^2 + 1$ not change the number or location of the x-intercepts?

16. Let $q(x) = 3x^2 - 7x - 6$.
   a. Graph q. Where do the roots of $q(x)$ appear to be located?
   b. Use the quadratic formula to solve $3x^2 - 7x - 6 = 0$.
   c. Use your result from Part b to factor $q(x)$.

17. Let $p(x) = k(x - 2)^2(x + 3)$, where $k \neq 0$.
   a. What are the roots of $p(x)$?
   b. If $p(-2) = 4$, find the value of k.

18. Let $q(m)$ be a polynomial, and let $q(m) = 0$ when $m = -3$, $m = 1$, and $m = 4$.
   a. Write a possible 3rd-degree equation for q.
   b. Write a possible equation for q that has degree 4.
   c. Graph your equations from Parts a and b. Describe any similarities and differences you see.

19. A manufacturer determines that n employees on a production line will produce $f(n)$ units per month where $f(n) = 80n^2 - 0.1n^4$.
   a. Factor the polynomial over the reals.
   b. Find the zeros of f.
   c. What do the zeros represent in this situation?
   d. Sketch a graph of f. Give a reasonable domain for this model.

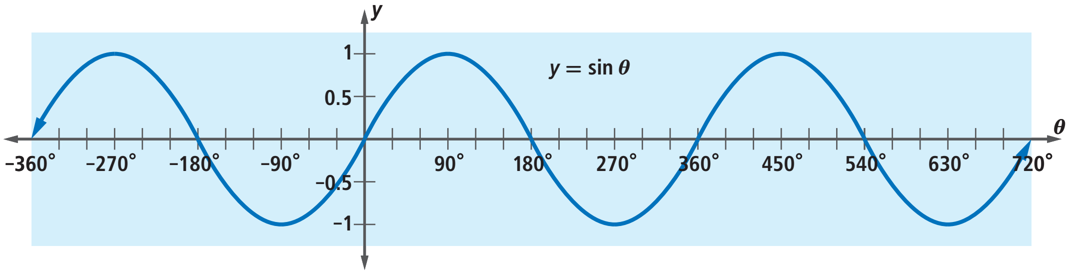
REVIEW

20. Consider the polynomial $p(x) = (2x - 3)(x^2 + 4)(10 - x)$. Without expanding the polynomial, find
   a. the leading term of $p(x)$ when written in standard form.
   b. the last term of $p(x)$ when written in standard form. **(Lesson 11-2)**

In 21–23, an expression is given. Tell whether the expression is a polynomial in x. If it is a polynomial in x, give its degree. If not, indicate why not. (Lesson 11-1)

21. $x^2y + xy^2 - \frac{1}{x}y^3$ 22. $\log(x^2y + xy^2)$ 23. $\sqrt{3}x^2y + \frac{\pi}{2}z$

In 24 and 25, use the graph below of the sine function.



24. What is the period of this function? **(Lesson 10-6)**

25. **Fill in the Blanks** As θ increases from 90° to 180°, the value of sin θ decreases from $\underline{\ ?\ }$ to $\underline{\ ?\ }$. **(Lesson 10-6)**

26. In the general compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, as the value of n increases while r and t are kept constant, the value of A gets closer and closer to what number? **(Lesson 9-3)**

27. a. **True or False** $\sqrt{12} \cdot \sqrt{3} = \sqrt{2} \cdot \sqrt{18}$. Justify your answer.
    b. **True or False** $\sqrt{-12} \cdot \sqrt{-3} = \sqrt{2} \cdot \sqrt{18}$. Justify your answer. **(Lessons 8-7, 8-5)**

28. A student, removing the bolts from the back of a large cabinet in a science lab, knew that it was easier to turn a bolt with a long wrench than with a short one. The student decided to investigate the force required with wrenches of various lengths, and obtained the data at the right. **(Lessons 2-7, 2-6, 2-2)**
    a. Graph these data points.
    b. Which variation equation is a better model for this situation, $F = \frac{k}{L}$ or $F = \frac{k}{L^2}$? Justify your answer.
    c. How much force would be required to turn one of these bolts with a 12-inch wrench?

Length of Wrench (in.) L	Force (lb) F
3	120
5	72
6	60
8	45
9	40

EXPLORATION

29. If $P(x) = (x - 5)(x - 3)^2(x + 1)^3$, 3 is called a *zero of multiplicity 2* and -1 is called a *zero of multiplicity 3*. Give equations for several different polynomial functions whose only real zeros are 5, 3, and -1, and graph them. How does the multiplicity of a zero affect the way the graph looks at the zero?

QY ANSWERS

1. $x - 5$

2. The factorization is $(x + 15)(x - 15)$. The roots are 15 and -15.