Lesson
11-3 Quick-and-Easy Factoring

Vocabulary

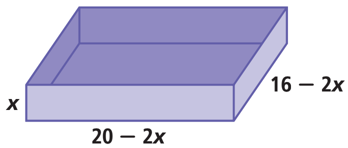
factoring a polynomial
factored form of a polynomial
greatest common monomial factor
prime polynomial, irreducible polynomial

BIG IDEA Some polynomials can be factored into polynomials of lower degree; several processes are available to find factors.

A polynomial is, by definition, a sum.

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x^1 + a_0,\ a_n \neq 0$$

However, polynomials are sometimes written as a product. For instance, recall the polynomial $4x^3 - 72x^2 + 320x$ from Lesson 11-2 that represents the volume of a box with sides of length x, $20 - 2x$, and $16 - 2x$. A *factored form* of this polynomial is $x(20 - 2x)(16 - 2x)$. By factoring out a 2 from each binomial in the factored form you can obtain another factored form. The expressions below are all equivalent.



$$4x^3 - 72x^2 + 320x \qquad (16 - 2x)(20 - 2x)x \qquad 4x(8 - x)(10 - x)$$

The process of rewriting a polynomial as a product of two or more factors is called **factoring the polynomial**, or writing the polynomial in **factored form**. Factoring undoes multiplication. Most factoring is based on three properties:

- Distributive Property (common monomial factoring)
- Special Factoring Patterns (difference of squares, etc.)
- Factor Theorem (for polynomials)

You may have used some of these properties to factor polynomials in earlier courses. The Factor Theorem is introduced in the next lesson. This lesson discusses how to apply the other two properties when factoring polynomials.

Mental Math

Let $\sin x = 0.15$. Find each of the following.

a. $\sin(-x)$

b. $\sin(180° - x)$

c. $\sin(180° + x)$

d. $\sin(360° + x)$

Common Monomial Factoring

Similar to the greatest common factor of a set of numbers, the **greatest common monomial factor** of the terms of a polynomial is the monomial with the greatest coefficient and highest degree that evenly divides all the terms of the polynomial. Common monomial factoring applies the Distributive Property.

Example 1

Factor $5x^3 - 15x^2$.

Solution Look for the greatest common monomial factor of the terms. The greatest common factor of 5 and 15 is 5. x^2 is the highest power of x that divides each term. So, $5x^2$ is the greatest common monomial factor of $5x^3$ and $-15x^2$. Now apply the Distributive Property.

$$5x^3 - 15x^2 = 5x^2(x - 3)$$

$5x^2(x - 3)$ is completely factored; it cannot be factored further.

Check Expand $5x^2(x - 3)$. $5x^2(x - 3) = 5x^2 \cdot x - 5x^2 \cdot 3 = 5x^3 - 15x^2$. It checks.

Special Factoring Patterns

CAS machines have built-in operations for factoring polynomials. In this Activity, you will use a CAS to discover some special factoring patterns.

Activity

MATERIALS CAS

Work with a partner.

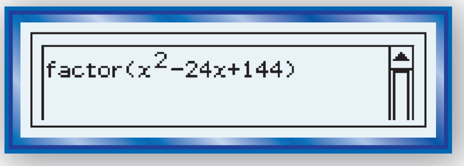
Step 1 Factor the following expressions on a CAS.

a. $x^2 - 24x + 144$ **b.** $x^2 - 49$

c. $x^2 - 64$ **d.** $x^2 + 8x + 16$

e. $4x^2 - 81$ **f.** $x^2 - 6x + 9$

g. $x^2 + 10x + 25$ **h.** $36x^2 - 1$



Step 2 Based on the results of Step 1, sort the polynomials into groups. Describe how you factor the polynomials in each group.

Step 3 Use your descriptions in Step 2 to factor the following.

a. $x^2 - y^2$ **b.** $x^2 - 2xy + y^2$ **c.** $x^2 + 2xy + y^2$

The three factoring relationships in the Activity are common enough to be worth knowing, and are summarized on the next page.

Difference of Squares Factoring Theorem

For all a and b,

$$a^2 - b^2 = (a + b)(a - b).$$

Binomial Square Factoring Theorem

For all a and b,

$$a^2 + 2ab + b^2 = (a + b)^2;$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

Be aware that many polynomials do not factor into binomials involving integers, or do not factor at all. For example, the sum of two squares, $a^2 + b^2$, cannot be factored over the set of polynomials with real coefficients.

GUIDED

Example 2

Factor each polynomial.

a. $9x^6 - 100y^2$

b. $p^2 - 18p + 81$

Solution

a. $9x^6 = (3x^3)^2$ and $100y^2 = (10y)^2$, so this polynomial is a difference of squares. Use difference of squares factoring.

$$9x^6 - 100y^2 = (\underline{\ ?\ })^2 - (\underline{\ ?\ })^2$$
$$= (\underline{\ ?\ } + \underline{\ ?\ })(\underline{\ ?\ } - \underline{\ ?\ })$$

b. This polynomial is in binomial square form with $a = p$ and $b = \pm 9$. In order for the signs to agree with the second Binomial Square Factoring pattern, you need $b = 9$. So,

$$p^2 - 18p + 81 = p^2 - 2 \cdot \underline{\ ?\ } \cdot \underline{\ ?\ } + \underline{\ ?\ }^2$$
$$= (\underline{\ \ ?\ \ })^2$$

Trial and Error

When there is no common monomial factor and the polynomial does not fit any of the special factoring patterns discussed above, a CAS can be used to factor. Additionally, some polynomials are simple enough that you can guess their factors and check by multiplying. Before CAS, some algebra students spent many hours learning strategies to guess more efficiently.

Example 3

Factor $x^2 - 2x - 15$.

Solution You want solutions to $x^2 - 2x - 15 = (x + \underline{\ ?\ })(x + \underline{\ ?\ })$.

Because –15 is the product of the two missing numbers on the right side, begin by considering the factors of –15. The factors are either –15 and 1, 15 and –1, 5 and –3, or –5 and 3. Try each combination by expanding until you find one that checks:

$(x - 15)(x + 1) = x^2 - 14x - 15$ Not correct

$(x + 15)(x - 1) = x^2 + 14x - 15$ Not correct

$(x + 5)(x - 3) = x^2 + 2x - 15$ Not correct

$(x - 5)(x + 3) = x^2 - 2x - 15$ Correct!

So, $x^2 - 2x - 15 = (x - 5)(x + 3)$.

Prime Polynomials

How do you know when a polynomial is factored completely? When you factor a number such as $12 = 2 \cdot 2 \cdot 3$, the number is factored completely when all the factors are prime. Similarly, a polynomial is factored completely when all the factors are *prime polynomials*. A polynomial is **prime**, or **irreducible**, over a set of numbers if it cannot be factored into polynomials of lower degree whose coefficients are in the set. Writing a polynomial as a product of factors with coefficients in a set is called *factoring over* that set.

Example 4

a. Is $x^2 - 14$ prime over the integers?

b. Is $x^2 - 14$ prime over the real numbers?

Solution

a. If $x^2 - 14$ factors over the integers, then you can find integers a and b so that $x^2 - 14 = (x + a)(x + b)$. Use trial and error to test values of a and b when $ab = -14$.

The integer factors of –14 are 1 and –14, –1 and 14, 2 and –7, or –2 and 7. Expand each combination.

$(x + 1)(x - 14) = x^2 - 13x - 14$

$(x - 1)(x + 14) = x^2 + 13x - 14$

$(x + 2)(x - 7) = x^2 - 5x - 14$

$(x - 2)(x + 7) = x^2 + 5x - 14$

None of these products expand to equal $x^2 - 14$.

So, $x^2 - 14$ is prime over the set of integers.

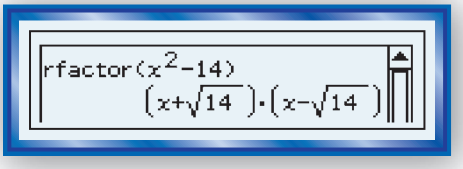
b. If $x^2 - 14$ factors over the real numbers, then you can find real numbers a and b so that $x^2 - 14 = (x + a)(x + b)$. Because a and b do not need to be integers, you can use Difference of Squares factoring.

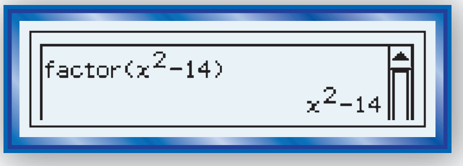
$$x^2 - 14 = x^2 - \left(\sqrt{14}\right)^2 = \left(x + \sqrt{14}\right)\left(x - \sqrt{14}\right)$$

This factorization shows that $x^2 - 14$ is not prime over the set of real numbers.

By default, most CAS machines factor over the rationals. However, a CAS will also factor over the real numbers, as shown below. Different machines may have different commands for factoring over the reals. Some machines factor over the reals when ",x" is entered after a polynomial in x. The machine output pictured at the left below uses an `rfactor` command to factor over the reals.

Note that when a polynomial is prime over a set, a CAS command to factor over that set will produce an output identical to the original polynomial. For example, the screen at the right below shows that $x^2 - 14$ is prime over the rationals.





Questions

COVERING THE IDEAS

1. **Fill in the Blanks** Copy and complete:
$21m^3n + 35m^2n - 14mn^2 = 7mn(\underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ })$

In 2 and 3, factor out the greatest common monomial. Check with a CAS.

2. $25y^4 - 50y$

3. $90x^3y^2 + 270xy^2 + 180x^2y^2$

In 4–9, a polynomial is given.
  a. Tell whether the polynomial is a binomial square, a difference of squares, or neither.
  b. Factor over the real numbers, if possible.

4. $a^2 - b^2$
5. $x^2 - 2xy + y^2$
6. $r^2 - 121$
7. $49x^2 - 144b^2$
8. $25b^2 - 70bc + 49c^2$
9. $y^2 + 25$

10. Check your solutions to Example 2 by expanding your answers.

11. **a.** Factor $12t^3 - 12t$ into linear factors.

b. Check by multiplying.

In 12 and 13, factor completely over the rational numbers and check your answer.

12. $64s^2 - 49$

13. $x^4 - 10x^2 + 9$

In 14–18, factor the polynomial completely over the rational numbers and state which factoring method(s) you used. If it is not factorable, write "prime over the rational numbers."

14. $3x^2 + 24x + 36$

15. $-1 - 10y - 25y^2$

16. $b^2 - 3b + 8$

17. $12x^3 - 75x$

18. **a.** $y^2 + 5y - 6$ **b.** $y^2 - 5y - 6$

c. $y^2 + 5y + 6$ **d.** $y^2 - 5y + 6$

19. **a.** Is $x^2 - 30$ prime over the integers? If not, factor it.

b. Is $x^2 - 30$ prime over the reals? If not, factor it.

APPLYING THE MATHEMATICS

20. $48^2 = 2304$. Use this information to factor $2303 = 48^2 - 1$.

21. **Multiple Choice** Which of the following is a perfect square trinomial?

A $16x^2 + 24x + 9$ B $q^4 - 16q^3 + 64$

C $t^2 - 64$ D $4x^2 + 16x + 25$

22. **a.** Write $x^4 - 16$ as the product of two binomials.

b. Write $x^4 - 64$ as the product of three binomials.

23. One factor of $12x^2 + x - 35$ is $(3x - 5)$. Find the other factor.

In 24 and 25, factor the polynomial by trial and error and check with a CAS.

24. $8z^2 + 6z + 1$

25. $6q^2 + 2q - 4$

In 26 and 27, a polynomial is given.

a. First factor out the greatest common monomial factor. Then complete the factorization.

b. Check by multiplying.

26. $242a^4b^2 - a^2b^4$

27. $5x^3 - 10x^2 - 175x$

28. **Multiple Choice** Which is a factorization of $a^2 + 100$ over the complex numbers?

A $(a + 10)(a + 10)$ B $(a + 10i)(a + 10i)$

C $(a + 10i)(a - 10i)$ D $(a + 10i)^2$

29. Consider the expression $\frac{4x^2 - 9}{2x^2 + x - 3}$.

a. Factor the numerator and denominator.

b. Because binomials are numbers, they may be multiplied and divided in the same way numbers are. Simplify your answer to Part a.

c. Using your factored expression in Part a, determine the domain of the function $f(x) = \frac{4x^2 - 9}{2x^2 + x - 3}$. Explain your reasoning.

REVIEW

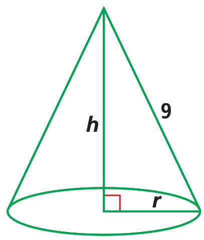
In 30 and 31, consider a closed rectangular box with dimensions $2h$, $h + 2$, and $2h + 3$. Write a polynomial in standard form for each measure.

30. $S(h)$, the surface area of the box (**Lesson 11-2**)

31. $V(h)$, the volume of the box (**Lesson 11-2**)

32. Give an example of a cubic binomial. (**Lesson 11-1**)

33. The lateral height of a cone is 9 centimeters and its height is h. (**Lessons 11-1, 1-7**)



a. Write a formula for the volume of the cone in terms of r and h.

b. Write a formula for the radius r of the cone in terms of h.

c. Substitute your expression for r in Part b into your formula in Part a.

d. **True or False** The volume of this cone is a polynomial function of h.

34. Write $\log_7 99 - \log_7 33$ as a logarithm of a single number. (**Lesson 9-9**)

35. Evaluate $\log_{13}\left(\frac{1}{13}\right)$. (**Lesson 9-7**)

EXPLORATION

36. Consider the polynomial $P(x) = \left(x - \frac{2}{3}\right)\left(x + \frac{3}{4}\right)$.

a. Write $P(x)$ in expanded form.

b. Use a CAS to factor the expanded form.

c. The answer to Part b does not look like the form you started with. What did the CAS do to the expanded form before it factored it?

d. Use the ideas in Parts a–c to factor $x^2 - \frac{5}{6}x + \frac{1}{6}$.