

Lesson

11-2

Multiplying
Polynomials

► **BIG IDEA** The product of two or more polynomials is a polynomial whose degree is the sum of the degrees of the factors.

Classifying Polynomials by the Number of Terms

In Lesson 11-1, you saw that polynomials can be classified by their degree. They can also be classified according to the number of terms they have after combining like terms. A **monomial** is a polynomial with one term, a **binomial** is a polynomial with two terms, and a **trinomial** is a polynomial with three terms. Below are some examples.

monomials: $-7, x^2, 3y^4$

binomials: $x^2 - 11, 3y^4 + y, 12a^5 + 4a^3$

trinomials: $x^2 - 5x + 6, 10y^6 - 9y^5 + 17y^2$

Notice that monomials, binomials, and trinomials can be of any degree. No special name is given to polynomials with more than three terms.

When a polynomial in one variable is added to or multiplied by a polynomial in another variable, the result is a polynomial in several variables. The **degree of a polynomial in several variables** is the largest sum of the exponents of the variables in any term. For instance, $x^3 + 8x^2y^3 + xy^2$ is a trinomial in x and y of degree 5. Notice that the sum of the exponents in the middle term is 5, while in both the first and last terms the sum of the exponents is 3.

The Extended Distributive Property

The product of a monomial and a binomial can be found using the Distributive Property, which says that for all numbers a , b , and c , $a(b + c) = ab + ac$. So, to multiply a monomial by a binomial, multiply the monomial by each term of the binomial and then add the products.

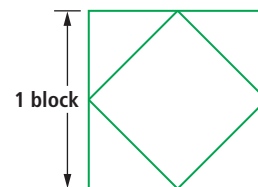
Repeated application of the Distributive Property allows you to find the product of any two polynomial factors. In general, if one polynomial has m terms and the second n terms, there will be mn terms in their product before combining like terms.

Vocabulary

monomial, binomial, trinomial
degree of a polynomial in
several variables

Mental Math

A square park is 1 block on a side. A person walks from the midpoint of one side to the midpoint of the next, and so on, until he returns to his starting point. How far has he walked?



► READING MATH

The prefixes *mono-*, *bi-*, *tri-*, and *poly-* mean “one,” “two,” “three,” and “many,” respectively. These prefixes are used in many common English words, such as monopoly, bicycle, and tricycle, and in geometric terms such as triangle and polyhedron.

Example 1

Expand $(2x^3 + 3x^2 - 2)(5x^2 + 4)$ and write your answer in standard form.

Solution 1 Expand on a CAS.

Solution 2 Use the Distributive Property by treating $(2x^3 + 3x^2 - 2)$ as a single unit.

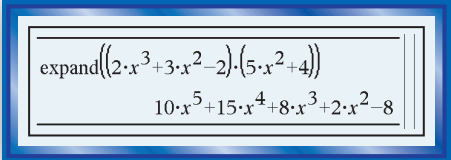
$$\begin{aligned}(2x^3 + 3x^2 - 2)(5x^2 + 4) \\ = (2x^3 + 3x^2 - 2)(5x^2) + (2x^3 + 3x^2 - 2)(4)\end{aligned}$$

Now use the Distributive Property to expand each product on the right side.

$$\begin{aligned} &= 2x^3 \cdot 5x^2 + 3x^2 \cdot 5x^2 + -2 \cdot 5x^2 + 2x^3 \cdot 4 + 3x^2 \cdot 4 + -2 \cdot 4 \\ &= 10x^5 + 15x^4 - 10x^2 + 8x^3 + 12x^2 - 8\end{aligned}$$

There are six terms. Combine like terms and write in standard form.

$$= 10x^5 + 15x^4 + 8x^3 + 2x^2 - 8$$



$$\begin{aligned}\text{expand}((2 \cdot x^3 + 3 \cdot x^2 - 2) \cdot (5 \cdot x^2 + 4)) \\ 10 \cdot x^5 + 15 \cdot x^4 + 8 \cdot x^3 + 2 \cdot x^2 - 8\end{aligned}$$

STOP QY

Notice that in Example 1 each of the terms of the trinomial $2x^3 + 3x^2 - 2$ is multiplied by each of the terms of the binomial $5x^2 + 4$. We call this generalization of the Distributive Property the *Extended Distributive Property*.

Extended Distributive Property

To multiply two polynomials, multiply each term in the first polynomial by each term in the second and add the products.

The Extended Distributive Property is applied several times when multiplying more than two polynomials. Because multiplication is associative and commutative, one way to multiply three polynomials is to start by multiplying any two of the polynomials and then multiplying their product by the remaining polynomial.

Used together, the Extended Distributive Property and the Associative Property of Multiplication let you multiply any number of polynomials in any order.

QY

Check Example 1 by letting $x = 2$ in both the given expression and the answer.

Example 2

- Find the volume of the large box by multiplying its dimensions.
- Find the volume of the large box by adding the volumes of each of the small boxes.
- Show that the answers to Parts a and b are equal.

Solution

- The box has width $Q + W$, height $R + S + T$, and depth D . Its volume is the product of its dimensions.

$$\text{Volume} = (Q + W)(R + S + T)D$$

- There are six small boxes, each with depth D . The volume of the big box is the sum of the volumes of the 6 smaller boxes (2 smaller boxes in each of 3 layers), from the top, left to right:

$$\text{Volume} = QRD + WRD + QSD + WSD + QTD + WTD$$

- The two expressions in Parts a and b must be equivalent because they represent the same volume. To show this, you can expand the product from Part a. Because of the Associative Property of Multiplication, either $Q + W$ can be multiplied by $R + S + T$ first, or $R + S + T$ can be multiplied by D first. We begin by multiplying by D first.

$$\text{Volume} = (Q + W)(RD + SD + TD)$$

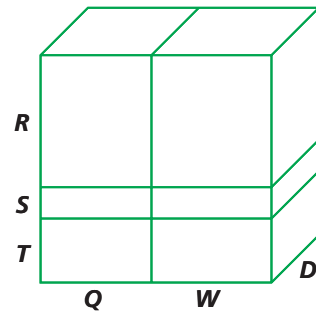
Now apply the Distributive Property, distributing $(Q + W)$ over the trinomial $RD + SD + TD$.

$$\text{Volume} = (Q + W)RD + (Q + W)SD + (Q + W)TD$$

Apply the Distributive Property again.

$$\text{Volume} = QRD + WRD + QSD + WSD + QTD + WTD$$

Notice how each term of the expanded form is the product of a term from $Q + W$, a term from $R + S + T$, and the term D .



Finding Terms of Products without Finding the Entire Product

In Example 1, you found that the product $(2x^3 + 3x^2 - 2)(5x^2 + 4)$ of two polynomials is equal to the polynomial $10x^5 + 15x^4 + 8x^3 + 2x^2 - 8$. All three polynomials were written in standard form. Notice the leading term $10x^5$ of the product is the product of the leading terms of the polynomial factors. Also, the last term -8 of the product is the product of the last terms of the factors.

In general, the leading term of the product of n polynomials written in standard form is the product of the leading terms of the polynomial factors, and the last term is the product of the last terms of the factors.

GUIDED

Example 3

Without expanding, find the leading term, the last term, and the coefficient of the term with x^3 of the product $(5x^2 + 2)(4x^2 + 8)(11x - 3)$ when written in standard form.

Solution The leading term is the product of the leading terms of the factors.

The leading term of the product is

$$\underline{\quad} \cdot \underline{\quad} \cdot \underline{\quad} = \underline{\quad}.$$

The last term is the product of the last terms of the factors.

The last term of the product is

$$\underline{\quad} \cdot \underline{\quad} \cdot \underline{\quad} = \underline{\quad}.$$

A term with x^3 will arise from multiplying $5x^2$ from the first factor, 8 from the second factor, and $11x$ from the third factor. The only other term with x^3 will arise from multiplying 2 from the first factor, $\underline{\quad}$ from the second factor, and $\underline{\quad}$ from the third factor. The first product is $\underline{\quad} x^3$; the second is $\underline{\quad} x^3$. So, after combining like terms, the term with x^3 in the product is $528x^3$.

Check Expand on a CAS and check the leading and last terms and the coefficient of x^3 in the product.

$$\text{expand}((5 \cdot x^2 + 2) \cdot (4 \cdot x^2 + 8) \cdot (11 \cdot x - 3))$$

Applications of Polynomials

A classic problem in mathematics is to find the maximum volume of an open box like the one in Example 4.

Example 4

A rectangular piece of cardboard measuring 16 inches by 20 inches is to be folded into an open box after cutting squares of side length x from each corner. Let $V(x)$ be the volume of the box.

- Write a polynomial formula for $V(x)$ in standard form.
- Use the graph of the function V to find the maximum possible volume.

(continued on next page)

Solution

- a. Draw a diagram.

When the cardboard is folded up, the dimensions of the box are $(20 - 2x)$ inches long by $(16 - 2x)$ inches wide by x inches high. The volume is the product of these dimensions.

$$V(x) = (20 - 2x)(16 - 2x)(x)$$

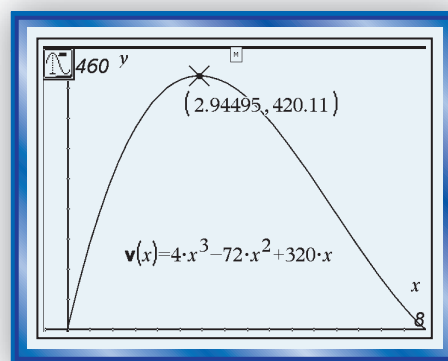
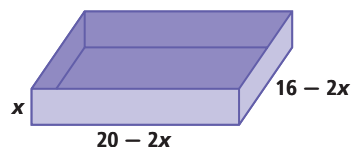
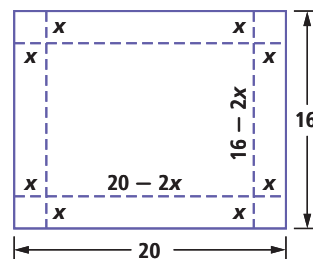
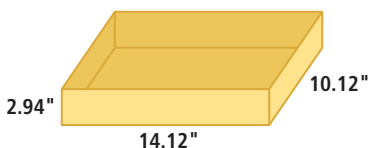
Use a CAS to expand this product. The CAS will automatically write the product in standard form.

$$\text{expand}((20-2x) \cdot (16-2x) \cdot x)$$

$$4 \cdot x^3 - 72 \cdot x^2 + 320 \cdot x$$

So $V(x) = 4x^3 - 72x^2 + 320x$.

- b. Because dimensions of a box are positive, $x > 0$, $20 - 2x > 0$ and $16 - 2x > 0$. Solving these inequalities for x , we have $x > 0$, $x < 10$, and $x < 8$, which means that $0 < x < 8$ is the largest domain for V in this situation. Graphing V over this domain shows that the largest possible volume is approximately 420.1 in^3 , which occurs when $x \approx 2.94 \text{ in}$. You can substitute this value for x to find that the box dimensions for this volume are 14.12 inches long by 10.12 inches wide by 2.94 inches high.



Questions

COVERING THE IDEAS

In 1–6, a polynomial is given.

- State whether the polynomial is a monomial, a binomial, a trinomial, or none of these.
 - Give its degree.
- $w^7 - w^5z^3 + \left(\frac{3}{2}\right)z^5$
 - $6a \cdot 2a \cdot \left(\frac{1}{3}\right)a$
 - $x^3 - x$
 - $4x^3 - 6x^2 - 5$
 - 3^2
 - $182x^4wt^2$
7. Give an example of a 5th-degree binomial in two variables.

In 8 and 9, multiply by hand and write in standard form.

8. $(5x^2 + \frac{1}{2}x - 2)(x + \frac{1}{5})$ 9. $(b^2 + 1)(2b - 3)(5b)$

10. Polynomial P is the product in standard form of

$$(12x^3 + 7)(3x^2 + \frac{3}{8}x - 1)(7x^2 - x).$$

Without expanding, write the first and last terms of P .

In 11 and 12, refer to Example 4.

11. a. Use $V(x) = 4x^3 - 72x^2 + 320x$ to complete the table below for $x = 0$ to 10.

x	1	2	3	4	5	6	7	8	9	10
$V(x)$	252	?	420	?	?	192	?	?	-36	?

b. For what integer value of x from 0 to 10 is $V(x)$ largest?

12. Give a reasonable domain for V if a new box has volume

$$V(x) = (15 - 2x)(18 - 2x)x.$$

APPLYING THE MATHEMATICS

13. Expand and write in standard form.

a. $(x^2 - 3x + 3)(x^2 + 3x + 3)$ b. $(x^2 - 4x + 4)(x^2 + 4x + 4)$

c. $(x^2 - 5x + 5)(x^2 + 5x + 5)$ d. $(x^2 - 6x + 6)(x^2 + 6x + 6)$

e. Based on your answers to Parts a–d, what do you predict the expanded form of $(x^2 - nx + n)(x^2 + nx + n)$ will be for any positive integer n ?

14. A box measures 12 inches by 18 inches by 10 inches. A y -foot long roll of wrapping paper is x feet wide. Assuming no overlap, how much wrapping paper will be left after wrapping the box?

15. A piece of material is in the shape of an equilateral triangle. Each side measures 15 inches. Kites with sides x and $x\sqrt{3}$ are cut from each corner. Then the flaps are folded up to form an open box.

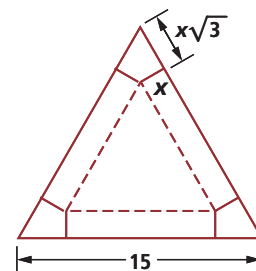
a. Write a formula for the volume $V(x)$ of the box as a product of polynomials.

b. Write $V(x)$ in standard form.

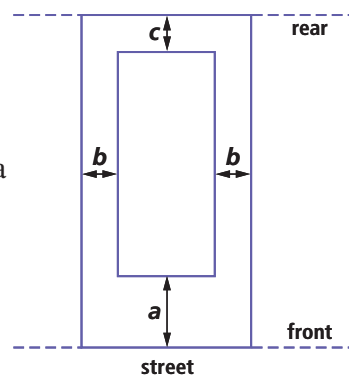
c. Find $V(4)$.

d. Find the maximum possible value of $V(x)$.

16. Melissa knows that $347 = 3 \cdot 10^2 + 4 \cdot 10 + 7$. Explain how Melissa can use what she knows together with the Extended Distributive Property to multiply any 3-digit number $a \cdot 10^2 + b \cdot 10 + c$ in base 10, where a , b , and c are single digits, by any 4-digit number in base 10.

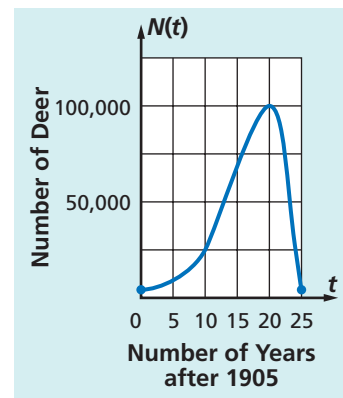


17. A town's zoning ordinance shows an aerial view of a lot like the one at the right. Distances a , b , and c are the minimum setbacks allowed to the street, the side lot lines, and the rear lot line, respectively. If a rectangular lot has 75 feet of frontage and is 150 feet deep, what is the maximum ground area possible for a one-story house in terms of a , b , and c ?
18. Find a monomial and a binomial whose product is $2p^2q + 4p$.
19. Find two binomials whose product is $2y^2 + 15y + 7$.
20. Find a binomial and a trinomial whose product is $3a^2 + 7ab + 2b^2 + 3a + b$.



REVIEW

21. Nancy invested different amounts at an APY of r . On the fifth anniversary of her initial investment her savings were $872x^5 + 690x^4 + 737x^3 + 398x^2 + 1152x + 650$ dollars, where $x = 1 + r$. (Lesson 11-1)
- What is the degree of the polynomial?
 - If Nancy invested at an APY of 5.125%, how much did Nancy have in the account on the fifth anniversary?
22. During the early part of the twentieth century, the deer population of the Kaibab Plateau in Arizona grew rapidly. Later, the increase in population depleted the food supply and the deer population declined quickly. The number $N(t)$ of deer from 1905 to 1930 is approximated by $N(t) = -0.125t^5 + 3.125t^4 + 4000$, where t is the time in years after 1905. This function is graphed at the right. (Lessons 11-1, 1-4)
- What is the degree of this polynomial function?
 - Estimate the deer population in 1905.
 - Estimate the deer population in 1930.
 - Over what time period was the deer population increasing?
23. a. Graph $c(x) = \cos x$ when $0^\circ \leq x \leq 360^\circ$. (Lesson 10-6)
b. For what values of x in this domain does $c(x) = 0$?
24. Solve $3x^2 - 16x - 64 = 0$. (Lesson 6-7)
25. Expand $(x + 7)^2$. (Lesson 6-1)



QY ANSWER

When $x = 2$, $(2x^3 + 3x^2 - 2) \cdot (5x^2 + 4) = (16 + 12 - 2) \cdot (20 + 4) = 624$.

When $x = 2$, $10x^5 + 15x^4 + 8x^3 + 2x^2 - 8 = 320 + 240 + 64 + 8 - 8 = 624$. It checks.

EXPLORATION

26. a. Find the products below.
 $(x - 1)(x + 1)$ $(x - 1)(x^2 + x + 1)$ $(x - 1)(x^3 + x^2 + x + 1)$
- b. State a general rule about polynomials whose product is $x^n - 1$, where $n = 2, 3, 4, 5, \dots$