Lesson 11-1 Introduction to Polynomials

Vocabulary

degree of a polynomial
term of a polynomial
polynomial in *x* of degree *n*
standard form of a polynomial
coefficients of a polynomial
leading coefficient
constant term, constant
polynomial function

BIG IDEA *Polynomials* are a common type of algebraic expression that arise from many kinds of situations, including those of multiple investments compounded over different lengths of time.

You are likely to have studied polynomials in a previous course. This lesson reviews some of the terminology that is used to describe them.

Mental Math

Give an example of the following or say that it does not exist.

a. a number without a multiplicative inverse

b. a matrix without an inverse

c. a relation without an inverse

d. a function whose inverse is not a function

Vocabulary Used with Polynomials

The expression

$$-0.001x^4 + 0.010x^3 + 0.060x^2 - 0.564x - 0.011$$

from the previous page is a *polynomial in the variable x*. When the polynomial is in only one variable, the largest exponent of the variable is the **degree of the polynomial**. The polynomial above has degree 4. The expressions $-0.001x^4$, $0.010x^3$, $0.060x^2$, $-0.564x$, and -0.011 are the **terms of the polynomial**. A polynomial is the sum of its terms.

Definition of Polynomial in *x* of Degree *n*

A **polynomial in *x* of degree *n*** is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$, where n is a nonnegative integer and $a_n \neq 0$.

The **standard form** of the general nth-degree polynomial is the one displayed in the definition. Notice that the terms are written in descending order of exponents. The numbers $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are the **coefficients of the polynomial**, with **leading coefficient** a_n. The number a_0 is the **constant term**, or simply the **constant**. For instance, the standard form of a 4th-degree polynomial is

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0.$$

It has coefficients a_4, a_3, a_2, a_1, and a_0, with leading coefficient a_4.

STOP QY1

QY1

What is the constant term of the 4th-degree polynomial $a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$?

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Example 1

a. List all the coefficients of $x^2 - 16x^4 + 3x^2 + 96$.

b. What are the degree and leading coefficient of $x^2 - 16x^4 + 3x^2 + 96$?

c. Rewrite the polynomial in standard form.

Solution

a. First, combine like terms.

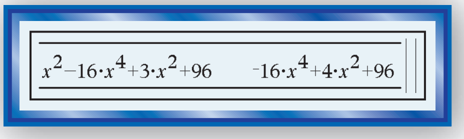
$$x^2 - 16x^4 + 3x^2 + 96 = (\underline{\ ?\ } + \underline{\ ?\ })x^2 - \underline{\ ?\ }x^4 + \underline{\ ?\ }$$
$$= \underline{\ ?\ }x^2 - \underline{\ ?\ }x^4 + \underline{\ ?\ }$$

So, by the definition on the previous page, $a_4 = \underline{\ ?\ }$, $a_3 = \underline{\ ?\ }$, $a_2 = \underline{\ ?\ }$, $a_1 = \underline{\ ?\ }$, and $a_0 = \underline{\ ?\ }$.

b. The largest exponent is $\underline{\ ?\ }$. So, the degree of the polynomial is $\underline{\ ?\ }$. The coefficient of the x^4 term is $\underline{\ ?\ }$. Thus, the leading coefficient is $\underline{\ ?\ }$.

c. In standard form, the terms are in descending order. So, the standard form of this polynomial is $\underline{\ ?\ }x^4 + \underline{\ ?\ }x^2 + \underline{\ ?\ }$.

Check Enter $x^2 - 16x^4 + 3x^2 + 96$ into a CAS. It automatically puts the polynomial in standard form.



Polynomials can be classified by their degree. Those of degree 1 through 5 have special names.

Degree	Polynomial Name	Example
1	Linear	$mx + b$
2	Quadratic	$ax^2 + bx + c$
3	Cubic	$ax^3 + bx^2 + cx + d$
4	Quartic	$ax^4 + bx^3 + cx^2 + dx + e$
5	Quintic	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

You can think of nonzero constants such as 5, $\pi + 2$, or a_0 as polynomials of degree 0. This is because a constant k can be written $k \cdot x^0$, which is a polynomial of degree 0. However, the constant 0 is not assigned a degree because its leading coefficient is zero.

Polynomial Functions and Graphs

A **polynomial function** is a function of the form $P: x \rightarrow P(x)$, where $P(x)$ is a polynomial. Polynomial functions of degree 1 are the linear functions and have graphs that are lines.

Polynomial functions of degree 2 are the quadratic functions and have graphs that are parabolas. For linear and quadratic functions, the coefficients of the polynomial help identify key points or properties of the graph such as slope, vertices, or intercepts. For polynomials of higher degree, the connection between the polynomial's coefficients and its graph is not as simple.

Example 2

Consider the polynomial function P with equation $P(x) = x^4 + 8x^3 + 20x^2 + 16x$.

a. **What is $P(-1)$?**

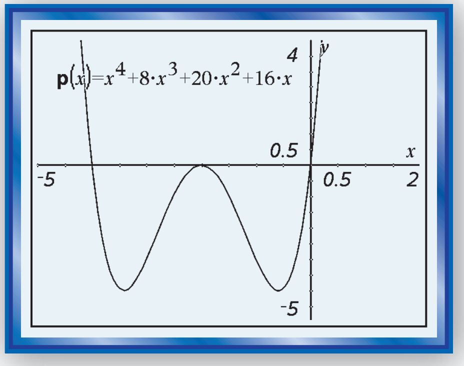
b. **Graph this function in the window $-5 \le x \le 2$, $-5 \le y \le 4$.**

Solution

a. Substitute –1 for x.

$$\begin{aligned} P(-1) &= (-1)^4 + 8(-1)^3 + 20(-1)^2 + 16(-1) \\ &= 1 - 8 + 20 - 16 \\ &= -3 \end{aligned}$$

b. A graph of P is shown at the right. The curve is related to the graph of $y = x^4$. The extra terms are responsible for the waves and translation off of the origin.



You will further explore relationships between coefficients and graphs in Lesson 11-5.

Operations on Polynomials and Polynomial Functions

Sums, differences, products, and powers of polynomials are themselves polynomials. The degree of the result of operations with polynomials depends on the degrees of the polynomials.

Activity 1

MATERIALS CAS

Let $f(x) = 2x^2 + 3x + 4$ and $g(x) = 5x^3 + 1$.

Step 1 Evaluate each expression. Write your answer in standard form and give the degree.

a. $f(x) + g(x)$ **b.** $f(x) - g(x)$

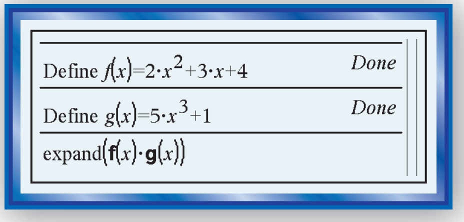
Step 2 Without multiplying the polynomials, predict

a. the degree of $f(x) \cdot g(x)$.

b. the leading coefficient of $f(x) \cdot g(x)$.

Step 3 **a.** Define the functions f and g on a CAS. Expand $f(x) \cdot g(x)$ and write the result in standard form.

b. Check your answers to Parts a and b of Step 2.



Step 4 Generalize the results of Steps 2 and 3:

The degree of the product of two polynomials is the __?__ of the degrees of the polynomials. The leading coefficient of the product of two polynomials is the __?__ of the leading coefficients of the polynomials.

The composite of two polynomial functions is a polynomial function.

Activity 2

MATERIALS CAS

Use the polynomials $f(x)$ and $g(x)$ from Activity 1.

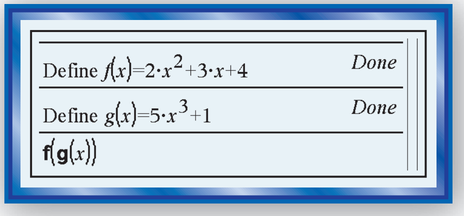
Step 1 Write an unsimplified expression for $f(g(x))$ and use it to predict

a. the degree of $f(g(x))$.

b. the leading coefficient of $f(g(x))$.

Step 2 **a.** Expand $f(g(x))$ on a CAS and write the result in standard form.

b. Check your answers to Parts a and b of Step 1.



Step 3 Generalize the results of Steps 1 and 2.

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Example 3

Let $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x - 5$. Predict

a. **the degree of $f(x) \cdot g(x)$.**

b. **the leading coefficient of $f(x) \cdot g(x)$.**

c. **the degree of $f(g(x))$.**

d. **the leading coefficient of $f(g(x))$.**

Solution

a. Because the degree of f is 2 and the degree of g is 1, the degree of $f(x) \cdot g(x)$ is __?__ + __?__ = __?__.

b. Because the leading coefficient of f is 3 and the leading coefficient of g is 4, the leading coefficient of $f(x) \cdot g(x)$ is __?__ · __?__ = __?__.

c. Because the degree of f is 2 and the degree of g is 1, the degree of $f(g(x))$ is __?__ · __?__ = __?__.

d. Because the leading coefficient of f is 3, the leading coefficient of g is 4, and the degree of f is 2, the leading coefficient of $f(g(x))$ is __?__ · __?__^{2} = __?__.

Savings and Polynomials

Recall that the total value over time of a single investment compounded at a constant rate can be described by an exponential function. If, instead, you have multiple investments compounded over different lengths of time, then the total value can be described by a polynomial function. Example 4 illustrates such a situation.

Example 4

Anita Loan and her family start saving for college when she graduates from eighth grade. At the end of each summer, they deposit money into a savings plan with an annual percentage yield (APY) of 5.8%. Anita is planning to go to college in the fall following her graduation from high school. How much money will be in the account when she leaves for college, if no other money is added or withdrawn?

End of Summer after:	Amount Deposited ($)
8th grade	1200
9th grade	850
10th grade	975
11th grade	1175
12th grade	1300

Solution The money deposited at the end of the summer after 8th grade earns interest for 4 years, so it is worth $\$1200(1.058)^4$ when Anita goes to college. Similarly, the amount deposited at the end of the summer after 9th grade is worth $\$850(1.058)^3$ because it only earns interest for 3 years. Adding the values at the end of each summer gives the total amount that will be in Anita's account. Notice that, because the last deposit does not earn any interest, the last term is not multiplied by a power of 1.058.

$$\underbrace{1200(1.058)^4}_{\text{End of summer after 8th grade}} + \underbrace{850(1.058)^3}_{\text{End of summer after 9th grade}} + \underbrace{975(1.058)^2}_{\text{End of summer after 10th grade}} + \underbrace{1175(1.058)^1}_{\text{End of summer after 11th grade}} + \underbrace{1300}_{\text{End of summer after 12th grade}}$$

Evaluating this expression shows that Anita will have about $6144.74 in her account when she leaves for college.

In Example 4, you could replace 1.058 with x. Then when Anita goes to college she will have (in dollars)

$$1200x^4 + 850x^3 + 975x^2 + 1175x + 1300.$$

If you let $x = 1 + r$, where r is the APY, evaluating this expression gives the amount in the account for any APY. You could use the expression to compare the total savings for different rates, or to compute the rate required to obtain a certain total. Since the first deposit earns interest for 4 years, the polynomial has degree 4.

STOP **QY2**

QY2

How much would Anita have if the APY in the account is 4.95%?

Questions

COVERING THE IDEAS

In 1–3, tell whether the expression is a polynomial. If it is, state its degree and its leading coefficient. If it is not a polynomial, explain why not.

1. $17 + 8y$
2. $14x^3 + 5x^{-1}$
3. $14x^3 + 12x^2 + 6x^5 + 3$
4. Refer to the definition of a polynomial of degree n. State each value for the polynomial $-x^6 + 16x^4 + 3x^3 + \frac{4x}{7} - 17$.
   a. n
   b. a_n
   c. a_{n-1}
   d. a_0
   e. a_1
   f. a_2
   g. a_4
5. Write the standard form of a quartic polynomial in the variable t.
6. **True or False** The number π is a polynomial.
7. Consider the polynomial function P with equation $P(x) = x^3 - 6x^2 + 3x + 10$.
   a. Evaluate $P(4)$.
   b. Graph P in the window $-2 \le x \le 6$, $-30 \le y \le 30$.

In 8 and 9, let $f(x) = 2x^4 - 1$ and $g(x) = \frac{3}{2}x^2 + 4x$. An expression is given.

a. Predict its degree.
b. Predict its leading coefficient.
c. Expand the expression and write the result in standard form.

8. $g(x) \cdot f(x)$
9. $g(f(x))$
10. Refer to Example 4. Suppose that in successive summers beginning after high school graduation, Javier put \$1250, \$750, \$2250, \$3500, and \$3300 into a bank account.
    a. Assume Javier goes to graduate school in the fall immediately after finishing 4 years of college and that the annual percentage yield is r. If no other money is added or withdrawn, how much is in his account when he goes to graduate school? Express your answer in terms of x, where $x = 1 + r$.
    b. Evaluate your answer to Part a when $r = 4.5\%$.
    c. What is the degree of the polynomial you found in Part a?

11. Suppose $f(x)$ is a polynomial of degree 3, and $g(x)$ is a polynomial of degree 5.
    a. What is the degree of $f(x) + g(x)$?
    b. What is the degree of $f(x) \cdot g(x)$?
    c. What is the degree of $f(g(x))$?

APPLYING THE MATHEMATICS

12. Let $q(x) = 4x^2 - 3$.

a. If $p(x) + q(x)$ has degree 3, what do you know about the degree of $p(x)$?

b. If $r(x) \cdot q(x)$ has degree 8, what do you know about the degree of $r(x)$?

c. If $s(x) \cdot q(x)$ has a leading coefficient of –28, what do you know about the leading coefficient of $s(x)$?

13. Recall the formula for the height h in feet of an object thrown upward: $h = -\frac{1}{2}gt^2 + v_0t + h_0$, where t is the number of seconds after being thrown, h_0 is the initial height in feet, v_0 is the initial velocity in $\frac{\text{ft}}{\text{sec}}$, and g is the acceleration due to gravity $(32\frac{\text{ft}}{\text{sec}^2}$ on Earth). This formula describes a polynomial function in t.

a. What is the degree of this polynomial?

b. What is the leading coefficient?

c. Suppose a ball is thrown upward from the ground with initial velocity $55\frac{\text{ft}}{\text{sec}}$. Find its height after 1.6 seconds.

14. Consider $f(x) = 4^x$ and $g(x) = x^4$.

a. Which of f or g is a polynomial function?

b. Which of f or g is an exponential function?

c. Explain how to tell the difference between an exponential function and a polynomial function.

15. The whole number 45,702 can be written as the polynomial function $P(x) = 4x^4 + 5x^3 + 7x^2 + 0x^1 + 2$, with $x = 10$.

a. Verify that $P(10) = 45{,}702$.

b. What is the base-10 value of the base-8 number 45,702?

16. In Hebrew, the number 18 stands for life and, for this reason, one custom is to give a child \$18 each year to save until his or her 18th birthday.

a. Suppose a child is given \$18 on each birthday (including the day he or she is born) until his or her 18th birthday, and that these gifts are put into an account with an annual yield of r. (No money is given on the 18th birthday itself.) Write a polynomial expression to give the total amount in the account on the child's 18th birthday. Let $x = 1 + r$.

b. Evaluate your answer to Part a for an APY of 4.2%.

REVIEW

17. Of the sequences A and B below, one is arithmetic, the other geometric. **(Lessons 7-5, 3-8, 3-6)**

A: 49, 7, 1, $\frac{1}{7}$, … B: 47, 36, 25, 14, …

a. Write the next two terms of each sequence.

b. Write an explicit formula for the geometric sequence.

c. Write an explicit formula for the arithmetic sequence.

d. Which sequence might model the successive maximum heights of a bouncing ball?

18. Solve this system of equations: $\begin{cases} r = 5t \\ s = r - 7 \\ t = r + 2s \end{cases}$. **(Lesson 5-3)**

19. A cat stalking a mouse creeps forward for 2 seconds at 0.5 $\frac{\text{ft}}{\text{sec}}$ then stops for 2 seconds. The cat springs forward at a rate of 10 $\frac{\text{ft}}{\text{sec}}$, but is stopped after 1 second by the refrigerator that the mouse ran under. Graph the situation, plotting time on the horizontal axis and distance on the vertical axis. **(Lesson 3-4)**

In 20 and 21, refer to the four equations below. (Lessons 2-6, 2-5, 2-4)

A $y = kx$ **B** $y = kx^2$ **C** $y = \frac{k}{x}$ **D** $y = \frac{k}{x^2}$

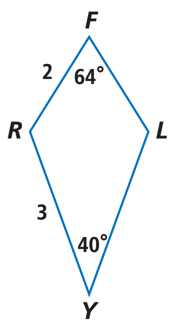
20. Which equations have graphs that are symmetric to the y-axis?

21. The graph of which equation is a parabola?

22. The weight of a body varies inversely with the square of the distance from the center of Earth. If Deja weighs 110 pounds on the surface of Earth, how much will she weigh in space, 6000 miles from the surface? (The radius of Earth is approximately 4000 miles.) **(Lesson 2-2)**

23. Refer to kite $FLYR$ at the right. **(Previous Course)**

a. Find LY. b. Find m$\angle R$.



EXPLORATION

24. Suppose $f(x)$ and $g(x)$ are polynomials of degree 4. Justify your answers with examples or proofs.

a. What are the possible degrees of $f(x) + g(x)$?

b. What are the possible degrees of $f(x) \cdot g(x)$?

c. What are the possible degrees of $f(g(x))$?

QY ANSWERS

1. a_0
2. \$6045.48