Lesson 6-1

Rate of Change

Vocabulary

rate of change

rate unit

BIG IDEA The rate at which a quantity changes can be determined either by computation or by looking at a graph.

What Is a Rate of Change?

Mr. and Mrs. Overjoyed had a healthy baby girl named Joy who weighed 7.5 pounds at birth. At the end of 4 months, the baby weighed 13.5 pounds. How fast did her weight change from birth to 4 months? To answer this question, we calculate the *rate of change* of Joy's weight, that is, how much she gained per month.

From 0 to 4 months

$$\frac{\text{change in weight}}{\text{change in age}} = \frac{13.5 \text{ lb} - 7.5 \text{ lb}}{4 \text{ mo} - 0 \text{ mo}} = \frac{6 \text{ lb}}{4 \text{ mo}} = 1.5 \frac{\text{lb}}{\text{mo}}$$

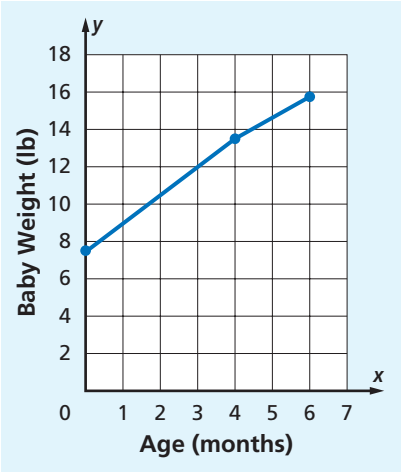
At age 6 months, Joy weighed 15.75 pounds. How fast did she grow from 4 months to 6 months? Use the same method to calculate the rate of change in her weight per month, from 4 months to 6 months.

From 4 to 6 months

$$\frac{\text{change in weight}}{\text{change in age}} = \frac{15.75 \text{ lb} - 13.5 \text{ lb}}{6 \text{ mo} - 4 \text{ mo}} = \frac{2.25 \text{ lb}}{2 \text{ mo}} = 1.125 \frac{\text{lb}}{\text{mo}}$$

Because $1.5 > 1.125$, Joy gained weight at a faster rate from 0 to 4 months than from 4 to 6 months.

These data points have been plotted at the right and connected to make a *line graph*. The **rate of change** measures how fast the graph goes up or down when reading from left to right. Since Joy's rate of change was greater for the 0 to 4 month period than for the 4 to 6 month period, the line segment connecting (0, 7.5) to (4, 13.5) is steeper than the line segment connecting (4, 13.5) to (6, 15.75).



Mental Math

A square chessboard has 64 squares, alternating black and white.

a. How many black squares are there?

b. How many white squares are there?

c. How many squares are on the edge of the board?

In 2003, there were 4,089,950 births in the United States.

Source: National Center for Health Statistics

Negative Rates of Change

Joy was gaining weight, so the rate of change was positive. But a rate of change can be negative.

Example 1

The population of Chicago from 1830 to 2000 is shown in the table and graph on page 325. Find the rate of change of the population of Chicago (in people per year) during the given time period, and describe how the rate is pictured on the graph.

a. 1890 to 1900 **b. 1970 to 1980**

Solutions

a. the rate of change, in people per year, from 1890 to 1900:

$$\frac{\text{change in population}}{\text{change in years}} = \frac{1{,}698{,}575 - 1{,}099{,}850}{1900 - 1890}$$

$$= \frac{598{,}725 \text{ people}}{10 \text{ yr}}$$

$$= 59{,}872.5 \frac{\text{people}}{\text{yr}}$$

Between 1890 and 1900 the population increased, so the rate of change is positive. The graph slants upward as you read the graph from left to right.

b. the rate of change between 1970 and 1980:

$$\frac{\text{change in population}}{\text{change in years}} = \frac{3{,}005{,}072 - 3{,}369{,}357}{1980 - 1970}$$

$$= \frac{-364{,}285 \text{ people}}{10 \text{ yr}}$$

$$= -36{,}428.5 \frac{\text{people}}{\text{yr}}$$

Between 1970 and 1980 the population decreased, so the rate of change is negative. In that interval, the graph slants downward as you read the graph from left to right.

When you read graphs in algebra, read them from left to right just as you would read a line in a book. A *positive rate of change* indicates that the graph slants *upward*. A *negative rate of change* indicates that the graph slants *downward*.

In both Joy's weight and Chicago's population, the *changes* are found by subtraction. The *rates* are found by division. So, you can calculate the rate of change between two points by dividing the difference in the y-coordinates by the difference in the x-coordinates. We use the subscripts $_1$ and $_2$ to identify the coordinates of the two points. For example, x_1 is read "x one" or "x sub one." The point (x_1, y_1) simply means the *first point*, while (x_2, y_2) means the *second point*.

Here is the general formula.

Rate of Change

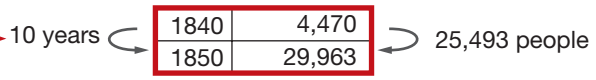
The rate of change between points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

Because every rate of change comes from division, the unit of a rate of change is a **rate unit.** In Example 1, a number of people is divided by a number of years. So, the unit of the rate of change is *people per year*, written as $\frac{\text{people}}{\text{year}}$ or people/year.

Using a Spreadsheet to Calculate Rate of Change

Spreadsheets and other table generators can be used to calculate rates of change. The spreadsheet below shows the years from 1830 to 2000 in column A, the population of Chicago in column B, and the rate of change of population for the previous decade in column C.

◇	A	B	C
1	Year	Population	Rate of change for previous decade
2	1830	100	
3	1840	4,470	437
4	1850	29,963	2549.3
5	1860	112,172	8220.9
6	1870	298,977	18680.5
7	1880	503,185	20420.8
8	1890	1,099,850	59666.5
9	1900	1,698,575	59872.5
10	1910	2,185,283	48670.8
11	1920	2,701,705	51642.2
12	1930	3,376,438	674773.3
13	1940	3,396,808	2037
14	1950	3,620,962	22415.4
15	1960	3,550,404	-7055.8
16	1970	3,369,357	-18104.7
17	1980	3,005,072	-36428.5
18	1990	2,783,726	-22134.6
19	2000	2,896,016	11229



Each rate of change is calculated using years and populations from two rows of the spreadsheet. Each formula in column C involves two subtractions, one to find the change in population and one to find the change in years. Then the population change is divided by the change in years. For example, C4 describes the change from 1840 to 1850.

The rate of change is $\frac{25{,}493 \text{ people}}{10 \text{ yr}} = 2{,}549.3 \frac{\text{people}}{\text{yr}}$. The formula to calculate this for C4 is = (B4–B3) / (A4–A3). Notice that cell C2 is empty because there is no population previous to 1830 in column B.

STOP QY

▶ QY

Chicago's population in 2005 was 2,842,518.

a. Find the rate of change of Chicago's population from 2000 to 2005.

b. The answer to Part a is negative. Explain what that means in the context of the problem.

A Zero Rate of Change

Sometimes quantities do not change over a certain interval. Then the rate of change is zero.

Example 2

The table below shows estimated attendance by the hour at a professional baseball game that went into extra innings.

a. During what time interval did the attendance not change?

b. Find the rate of change of attendance during that time.

Time	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	7 P.M.
Attendance	1,200	18,400	23,200	23,200	23,200	20,100	2,000

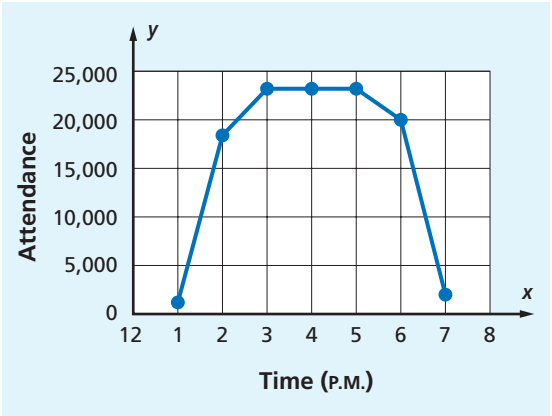
Fans watch the first pitch at the first regular season game at Pacific Bell Park in San Francisco in 2000.

Solutions

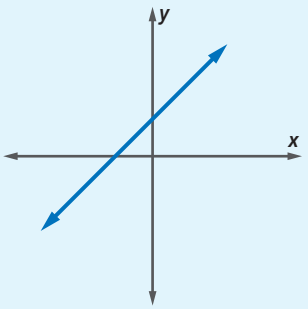
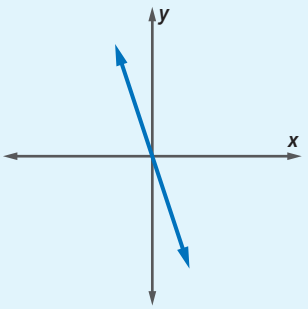
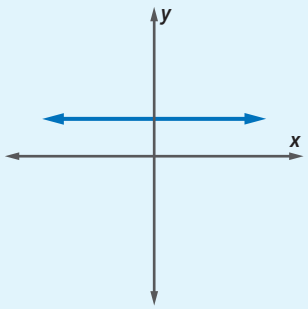
a. From 3:00 pm to 5:00 pm the attendance did not change. Notice this is associated with a horizontal line segment on the graph at the right.

b. Use the coordinates (3, 23,200) and (5, 23,200) to calculate the rate of change.

$$\text{rate of change} = \frac{23{,}200 - 23{,}200}{5 - 3} = \frac{0 \text{ people}}{2 \text{ hours}} = 0 \frac{\text{people}}{\text{hour}}$$



In general, a rate of change of 0 corresponds to a horizontal segment on the graph. The table on page 330 summarizes the relationship between rate of change and the graph.

Situation	Constant Increase	Constant Decrease	No Change
Rate of Change	positive	negative	zero
Slant (left to right)	upward	downward	horizontal
Sketch of Graph			

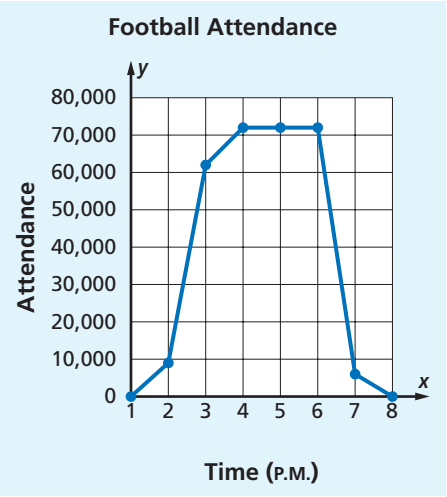
Questions

COVERING THE IDEAS

1. **Fill in the Blanks** A rate of change is given in dollars per hour. It is found by dividing change in ___?___ by change in ___?___.
2. The table at the right shows Jack's weight from birth to age 2.
   a. Make a line graph for this data.
   b. Which is steeper, the segment joining (0, 8) to (4, 15) or the segment joining (6, 19) to (12, 24)?
   c. Which has the greater rate of change, the segment joining (0, 8) to (4, 15) or the segment joining (6, 19) to (12, 24)?
   d. Explain why we would not expect the rate of change of Jack's weight to be negative during his childhood.
   e. Was the rate of change in his weight greater from ages 6 to 12 months or from 12 to 18 months?
   f. What is the rate of change in his weight from birth to 24 months?
3. Refer to the graph and table of the population of Chicago on page 325.
   a. In which decade did Chicago's population grow the fastest?
   b. In which 20-year period did the population of Chicago grow the fastest?
   c. In which decade did the population of Chicago decline the most?
   d. Use the spreadsheet of the population of Chicago on page 328 to find the formula that created cell C10.

Age	Weight (lb)
birth	8
4 mo	15
6 mo	19
12 mo	24
18 mo	27
24 mo	29

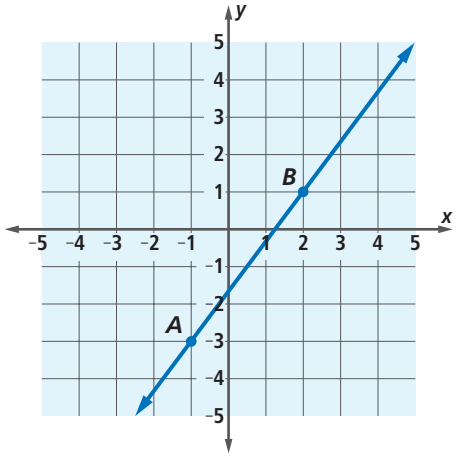
In 4–8, refer to the graph at the right of attendance at a football game.



4. Identify all one-hour time periods where the rate of change is negative.
5. Did attendance increase or decrease between 2:00 P.M. and 3:00 P.M.?
6. Calculate the rate of change in attendance from 5:00 P.M. to 6:00 P.M.
7. Did the attendance increase, decrease, or stay the same from 6:00 P.M. to 7:00 P.M.?
8. When do you think the game started and ended? Explain your answer.
9. Evaluate the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for the points (–6, 9) and (–3, 7).

APPLYING THE MATHEMATICS

10. Find the rate of change from point A to point B on the graph at the right.



11. Refer to the table below.

Hours After Midnight	Height of Tide Above Sea Level (m)
0	0.9
6	0.6
12	–0.2
18	0.5
24	0.9

   a. Calculate the rate of change from 12 hours to 18 hours after midnight.
   b. Is the rate of change between 6 and 12 hours after midnight positive, negative, or zero?

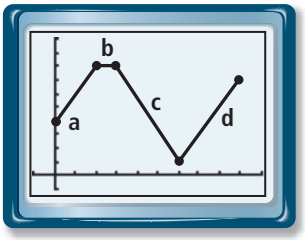
12. Refer to the table at the right of the population of Hong Kong, one of the most densely populated places in the world.
   a. In which time period was the rate of population growth per year the greatest?
   b. Hong Kong has an area of about 425 square miles. In 1999, Hong Kong had about how many people per square mile?

Year	Population
2006	6,940,432
1999	6,840,600
1991	5,647,114
1986	5,395,997
1981	4,986,560
1976	4,439,250

Source: *Advertising Age*

13. The Beier family had $2,324 in a checking account on December 1st and $490 in the same account two weeks later. In this time period, what was the rate of change of the amount of money in their account per day?

14. A bulldozer is working on a highway. The graph at the right shows the distance between the bulldozer and a fixed point on the highway measured over time. Use the graph to complete the table.



Section of Graph	Rate of Change (positive, negative, or zero)	Bulldozer's Movement (towards point, away from point, standing still)
a	?	away from point
b	?	?
c	?	?
d	positive	?

REVIEW

15. **True or False** The rate *9 apple pies for 15 people* is equal to the rate $\frac{3}{5}$ *apple pie for 1 person*. Explain your answer. **(Lesson 5-1)**

16. **a.** Graph $y = 2x + 5$ and $y = -2x + 5$ on the same set of axes.
b. Where do the two lines intersect? **(Lessons 4-3, 3-1)**

17. A numismatist (coin collector) has a collection of 850 coins. If the collection grows at 48 coins per year, how many coins will the numismatist have in x years? **(Lesson 1-2)**

18. Suppose $2x + 5y = 10$. **(Lesson 1-1)**
a. Find the value of y when $x = 0$.
b. Find the value of y when $x = 3$.
c. Find the value of x when $y = 0$.
d. Find the value of x when $y = -4$.

The collecting of coins is one of the oldest hobbies in the world.

Source: *Encyclopædia Britannica*

EXPLORATION

19. Find the population of the town you live in for the years 1940, 1950, 1960, 1970, 1980, 1990, and 2000. Put this information into a spreadsheet and calculate the rate of change of population for each decade.
a. In which 10-year period was the rate of growth greatest?
b. In which decades was there negative growth? Positive growth? Zero growth?
c. In which 20-year period was the rate of growth greatest?

QY ANSWERS

a. $-10{,}699.6 \frac{\text{people}}{\text{yr}}$

b. Chicago's population decreased from 2000 to 2005.